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Wave Functions for Galilean Particles

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We give a description of Galilean particles in terms of geometric quantization, a geometric correspondence from quantum states in the preceding sense to wave functions in the ordinary quantum mechanical sense, and explicit computations in some cases that lead to Schrödinger and Pauli equations.

1. INTRODUCTION

This paper should be placed in the general framework of the description of nonrelativistic particles. The search for such a description is of some interest from both the epistemological and the pedagogical point of view. The study of this problem leads in particular to a clearcut distinction between the specifically relativistic features of relativistic quantum mechanics and those that equally follow from a nonrelativistic quantum theory. One of the interesting conclusions is that nonrelativistic particles seem to possess intrinsic moments with the same values as their relativistic counterparts. In particular, the spin magnetic moment with its Landé factor $g = 2$ is not a relativistic property. This has been proved by Lévy-Leblond [12]. Other applications of the theory are given in ref. 13.

The usual relativistic quantum mechanics leads to many wave equations, each for a different kind of particle. In particular, one deals with Klein–Gordon, Dirac, Maxwell, Weyl, gravitino, or Penrose wave equations. Each of these equations was derived independently, but the recognition of their spaces of solutions as spaces of representation of the Poincaré group leads to a unification. In fact a group-theoretic study of wave equations was made by Bargmann and Wigner [3] based on the previous classification of the representations of Poincaré group made by Wigner [16].

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On the other hand, it is well known that geometric quantization has its origins in a method to obtain group representations, the so-called orbit method, first used by Kirillov [9] to obtain all unitary representations of nilpotent Lie groups. That method was extended by Auslander and Kostant [1] to obtain all unitary representations of Type I solvable Lie groups. These methods are intimately related to geometric quantization (Kostant–Souriau theory [11, 14]). Then it is natural to ask for the possibility of using this point of view to obtain the different wave equations. This has been accomplished in many cases in ref. 14. [14]. These results were completed in the relativistic case in ref. 8.

In the present paper, similar results are proved for the Bargmann group, which is a central extension of the Galilei group.

As usual in geometric quantization, each coadjoint orbit corresponds to a different kind of particle. Some of the orbits, the so-called quantizable ones, are the base space of a natural principal circle bundle. The quantum states of the corresponding particle are the sections of the associated Hermitian line bundle, which satisfies a certain invariance condition. We prove that these sections are in a one-to-one correspondence with the unrestricted sections of another Hermitian line bundle.

The correspondence of the later sections with wave functions is made in two steps. First, we establish a correspondence of the sections with those of another line bundle and then we immerse this later bundle in a trivial one. The sections under consideration are thus in a one-to-one correspondence with vector-valued functions on the base space. These functions gives rise by integration to the wave functions of the corresponding particle.

Some particular cases are considered and we see that in one case the wave functions compose a wide family of solutions of the Schrödinger equation. In another case, they compose a family of solutions of the Pauli equations. These equations were first proposed as a nonrelativistic limit of the Dirac equation, but here they appear as Galilei-invariant equations. This invariance was first remarked by Levy-Leblond [12].

2. GALILEAN RELATIVITY GROUPS

The Galilei group is the differentiable manifold $\mathcal{G} = O(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, with the group law given by

$$(A, b, c, e) \star (A', b', c', e') = (AA', Ab' + b, Ac' + be' + c, e + e')$$

for all $(A, b, c, e), (A', b', c', e') \in \mathcal{G}$. Its connected component of the identity is $\mathcal{G}_0 = SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$.

To each $(A, b, c, e) \in \mathcal{G}$ there is associated the diffeomorphism of $\mathbb{R}^4$ defined by sending (x, y, z, t) to (x', y', z', t'), where

$$\bar{x}' = A\bar{x} + bt + c \tag{2.1}$$

$$t' = t + e \tag{2.2}$$

with $\bar{x} = {}^t(x, y, z)$ and $\bar{x}' = {}^t(x', y', z')$. Here ${}^tM$ means the transpose of the matrix M. This association defines the usual action on the left of $\mathcal{G}$ on $\mathbb{R}^4$.

The Lie algebra of $\mathcal{G}$, $\underline{\mathcal{G}}$, can be identified as usual with the tangent space to the identity element, i.e., to $o(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$.

On the other hand, $o(3)$ can be identified to $\mathbb{R}^3$ by associating $b = (f, g, h)$ to

$$\hat{b} = \begin{pmatrix} 0 & -h & g \\ h & 0 & -f \\ -g & f & 0 \end{pmatrix} \tag{2.3}$$

Thus, the Lie bracket in $o(3)$ becomes the usual cross-product in $\mathbb{R}^3$. As a consequence, we can identify $\underline{\mathcal{G}}$ with $(\mathbb{R}^3)^3 \times \mathbb{R}$.

We use the basis of $\underline{\mathcal{G}}$ composed by the elements $Z_G^1, \ldots, Z_G^{10}$, defined by $Z_G^i = (e_i, 0, 0, 0)$, $Z_G^{3+i} = (0, e_i, 0, , 0)$, $Z_G^{6+i} = (0, 0, e_i, 0)$, $Z_G^{10} = (0, 0, 0, 1)$, $i = 1, 2, 3$, where e_1, e_2, e_3 are the elements of the canonical basis of $\mathbb{R}^3$. The elements of the dual basis are denoted by $(Z_G)_i^*$, $i = 1, \ldots, 10$.

The Bargmann group [2] is a central extension of $\mathcal{G}$ by $\mathbb{R}$. It consists of the manifold $O(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ provided with the group law given by

$$\begin{aligned}
&(A, b, c, e, a) \star (A', b', c', e', a') \\
&\quad = \left(AA', Ab' + b, Ac' + be' + c, e + e', a + a' + {}^tbAc' + \frac{b^2}{2}e'\right)
\end{aligned}$$

for all $(A, b, c, e, a), (A', b', c', e', a') \in O(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$. Its connected component of the identity is $SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$, which will be denoted in what follows by $\mathcal{B}$.

The map from the Bargmann group onto the Galilei group given by the canonical projection onto the first four components is a homomorphism whose kernel is in the center and is isomorphic to $\mathbb{R}$ under the map defined by sending $t \in \mathbb{R}$ to $(I, 0, 0, 0, t)$. The tangent vector at $t = 0$ to this one-parameter subgroup is denoted by Z^{11}. The homomorphism has a section defined by sending (A, b, c, e) to $(A, b, c, e, 0)$. The tangent map to this section at the identity provides us with an injective (not homomorphic) map from Galilei Lie algebra $\underline{\mathcal{G}}$ into the Bargmann Lie algebra $\underline{\mathcal{B}}$. The elements corresponding to the Z_G^i will be denoted by Z^i.

Now, let us consider the closed subgroup of $GL(5, \mathbb{R})$, Γ, composed by the matrices

$$\begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

with $A \in O(3)$, b, $c \in \mathbb{R}^3$, $e \in \mathbb{R}$,

The map ϕ from $\mathcal{G}$ onto Γ defined by sending $(A, b, c, e) \in \mathcal{G}$ to (2.4) is an isomorphism of Lie groups. In what follows we identify these Lie groups by means of ϕ.

We also consider the closed subgroup of $GL(6, \mathbb{R})$, Δ, composed by the matrices

$$\begin{pmatrix} A & b & c & 0 \\ 0 & 1 & e & 0 \\ 0 & 0 & 1 & 0 \\ {}^t bA & b^2/2 & a & 1 \end{pmatrix} \tag{2.5}$$

with $A \in O(3)$, b, c, $\in \mathbb{R}^3$, $e \in \mathbb{R}$,

The map δ from the Bargmann group onto Δ defined by sending (A, b, c, e, a) to (2.5) is an isomorphism of Lie groups. In what follows we also identify these Lie groups by means of δ.

These representations can be used to determine the coadjoint representations of Γ and Δ. For example, to evaluate the matrix of $Ad_{(A,b,c,e,a)^{-1}}$ in the basis composed by the Z^i, one can proceed as follows.

The element of the Lie algebra of Δ whose components in the basis $(d\delta \cdot Z^1, \ldots, d\delta \cdot Z^{11})$ are $(\omega^1, \omega^2, \omega^3, \beta^1, \beta^2, \beta^3, \gamma^1, \gamma^2, \gamma^3, \varepsilon, \alpha)$ is

$$\begin{pmatrix} \hat{\omega} & \beta & \gamma & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ {}^t\beta & 0 & \alpha & 0 \end{pmatrix} \tag{2.6}$$

where $\beta = {}^t(\beta^1, \beta^2, \beta^3)$ and $\gamma = {}^t(\gamma^1, \gamma^2, \gamma^3)$, and its image under $Ad_{(A,b,c,e,a)^{-1}}$ is

$$\delta((A, b, c, e, a)^{-1}) \begin{pmatrix} \hat{\omega} & \beta & \gamma & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ {}^t\beta & 0 & \alpha & 0 \end{pmatrix} \delta((A, b, c, e, a))$$

By direct computation and regrouping of terms in ω^i, β^i, γ^i, ε, α one finds the sought-for matrix. Its transpose gives the matrix of $Ad^*_{(A,b,c,e,a)}$ in the basis dual of the $(Z^i\colon i = 1, \ldots, 11)$, $(\overset{*}{Z}_i\colon i = 1, \ldots, 11)$, and is given by

$$\begin{pmatrix} A & \hat{b}A & \hat{c}A & 0 & b \times c \\ 0 & A & eA & 0 & c - eb \\ 0 & 0 & A & 0 & -b \\ 0 & 0 & -{}^t bA & 1 & b^2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

This gives us the coadjoint representation.

An explicit expression for the coadjoint representation of the Galilei group follows from the preceding matrix: the matrix of $Ad^*_{(A,b,c,e)}$ in the basis $((Z_G)_i^*: i = 1, \ldots, 10)$ is

$$\begin{pmatrix} A & \hat{b}A & \hat{c}A & 0 \\ 0 & A & eA & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & -{}^t bA & 1 \end{pmatrix} \tag{2.8}$$

The connected components of the identity of Galilei and Bargmann groups are obtained by substitution of $O(3)$ by $SO(3)$. If one restricts consideration to these Lie groups, the preceding formulas concerning the coadjoint representation remain valid.

In this paper we consider the universal covering group of the connected component of the identity of the Bargmann group, which we introduce as follows.

Let us denote by q: $SU(2) \to SO(3)$ the natural covering map. This map is defined as follows.

Let $H_0(2)$ be the real vector space composed of the 2×2 traceless Hermitian matrices and let h: $\mathbb{R}^3 \to H_0(2)$ be the isomorphism of real vector spaces given by

$$h(x^1, x^2, x^3) = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}$$

If $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$, we have

$$\|(x^1, x^2, x^3)\|^2 = -Det\, h(x^1, x^2, x^3) \tag{2.9}$$

For each $A \in SU(2)$ we define a diffeomorphism of $H_0(2)$, Φ_A, by means of $\Phi_A(H) = AHA^*$ for all $H \in H_0(2)$, where A^* is the transpose conjugate of A.

The map q is defined by sending A to the matrix of $h^{-1} \circ \Phi_A \circ h$ in the canonical basis of $\mathbb{R}^3$. Since Φ_A preserves the determinant, it follows from (2.9) that the image of q is in $O(3)$. As a consequence of the fact that $SU(2)$ is connected (it is diffeomorphic to the sphere $\mathbb{S}^3$), the image of q is in $SO(3)$.

The map q is a homomorphism whose kernel is $\pm I$. Thus, since the dimensions of $SU(2)$ and $SO(3)$ coincide, q is a twofold covering map. The

Lie group $SU(2)$ being simply connected, it is the universal covering group of $SO(3)$.

Let σ_1, σ_2, σ_3 be the Pauli matrices, i.e.,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively.

A basis of $su(2)$ is $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ and one sees by direct computation that $\{dq \cdot i\sigma_1, dq \cdot i\sigma_2, dq \cdot i\sigma_3\}$ is the basis of $so(3)$, $\{-2\hat{e}_1, -2\hat{e}_2, -2\hat{e}_3\}$ [cf. equation (2.3)].

Now we define a map Q from the set $\tilde{\mathcal{B}} = SU(2) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ onto $\mathcal{B} = SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ by means of $Q(A, b, c, e, a) = (q(A), b, c, e, a)$. When one considers on these sets its natural manifold structure, Q becomes a covering map, so that $\tilde{\mathcal{B}}$ is the universal covering manifold of $\mathcal{B}$.

In $\tilde{\mathcal{B}}$ we consider the group structure given by

$$\begin{aligned} &(A, b, c, e, a) \perp (A', b', c', e', a') \\ &\quad = \Big(AA', q(A)b' + b, q(A)c' + be' + c, e + e', a + a' \\ &\qquad + {}^t b q(A) c' + \frac{b^2}{2} e'\Big) \end{aligned}$$

Provided with this group structure, $\tilde{\mathcal{B}}$ becomes a Lie group, and then the universal covering group of $\mathcal{B}$.

The Lie algebra isomorphism dQ^{-1} sends the basis composed by the Z^i to a basis of the Lie algebra $\underline{\tilde{\mathcal{B}}}$ of $\tilde{\mathcal{B}}$, whose elements, or its opposed, receive the following designations:

$$\begin{aligned} l^i &= dQ^{-1} \cdot Z^i \\ g^i &= -dQ^{-1} \cdot Z^{i+3} \\ p^i &= dQ^{-1} \cdot Z^{i+6} \\ E &= -dQ^{-1} \cdot Z^{10} \\ \underline{m} &= -dQ^{-1} \cdot Z^{11} \end{aligned} \tag{2.10}$$

for all $i = 1, 2, 3$. The reason for these denominations is that these elements of the Lie algebra will represent, in a suitable sense that will be specified in Section 3, the dynamical variables angular momentum, linear momentum, energy, and mass, respectively.

The dual of the basis composed by the $dQ^{-1} \cdot Z^i$ is given by $\{{}^t(dQ) \cdot \overset{*}{Z}{}^1, \ldots, {}^t(dQ) \cdot \overset{*}{Z}{}^{11}\}$. The matrix of $Ad^*_{(U,b,c,e,a)}$ in this basis is

$$\begin{pmatrix} q(U) & \hat{b}q(U) & \hat{c}q(U) & 0 & b \times c \\ 0 & q(U) & eq(U) & 0 & c - eb \\ 0 & 0 & q(U) & 0 & -b \\ 0 & 0 & -{}^tbq(U) & 1 & b^2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.11}$$

Then, its matrix in the basis dual of the $(l^1, l^2, l^3, g^1, g^2, g^3, p^1, p^2, p^3, E, \underline{m})$ is

$$\begin{pmatrix} q(U) & -\hat{b}q(U) & \hat{c}q(U) & 0 & c \times b \\ 0 & q(U) & -eq(U) & 0 & c - eb \\ 0 & 0 & q(U) & 0 & b \\ 0 & 0 & {}^tbq(U) & 1 & b^2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.12}$$

The group $\tilde{\mathcal{B}}$ can also be described as a semidirect product. Let us recall some terminology concerning semidirect products of Lie groups.

Let H and K be Lie groups and s be a homomorphism of K into the Lie group of automorphisms of H. In $K \times H$ we consider the group law given by

$$(k, h) * (k', h') = (kk', hs_k(h'))$$

where s_k stands for the automorphism of H associated to $k \in K$ by s, and the product manifold structure. Thus $K \times H$ becomes a Lie group that is called the semidirect product of K and H and is denoted by $K \times_s H$. If $s_k = Id_H$ for all $k \in K$, we write simply $K \times H$ instead of $K \times_s H$, and this Lie group is called the direct product of K and H.

Now we define $\tilde{\mathcal{B}}_0 = SU(2) \times_s \mathbb{R}^3$, where s is the map defined by sending $A \in SU(2)$ to $h^{-1} \circ \Phi_A \circ h$. Thus the group law in $\tilde{\mathcal{B}}_0$ can be written as $(A, b) * (A', b') = (AA', q(A)b' + b)$.

Let r be the homomorphism of $\tilde{\mathcal{B}}_0$ into the group of authomorphisms of $\mathbb{R}^5$ defined by sending $(A, b) \in \tilde{\mathcal{B}}_0$ to

$$r(A, b)\colon \quad (c, e, a) \in \mathbb{R}^5 \to \left(q(A)\, c + be, e, {}^tbq(A)\, c + \frac{b^2}{2}\, e + a\right) \in \mathbb{R}^5$$

where c consists of the first three coordinates of (c, e, a).

Obviously, $\tilde{\mathcal{B}}$ is isomorphic to $\tilde{\mathcal{B}}_0 \times_r \mathbb{R}^5$, and these Lie groups will be identified in the following.

In the remainder of this paper we write Ax to means $q(A)x$, for all $A \in SU(2)$, $x \in \mathbb{R}^3$.

3. CLASSICAL STATE SPACE

This paper is based on the following physical hypothesis. The Galilean particles are divided into classes, each class corresponding, according to geometric quantization [14], to a coadjoint orbit of the relativity group of the theory. In the present case the relativity group is the universal covering group $\tilde{\mathcal{B}}$ of the Bargmann group (cf. Section 2).

Each element of a coadjoint orbit is interpreted as being a movement of the corresponding particle. The coadjoint orbit itself is called *movement space*.

In this paper we consider as a *classical state space* of a Galilean particle any orbit of $\tilde{\mathcal{B}}$ in $\mathbb{R}^4 \times \underline{\tilde{\mathcal{B}}}^*$ where the action is the product of the coadjoint one onto the second factor and the action on $\mathbb{R}^4$ given by

$$L * x = \pi(Q(L)) \cdot x \tag{3.1}$$

for all $L \in \tilde{\mathcal{B}}$, $x \in \mathbb{R}^4$, where Q is the covering map, π the projection onto the Galilei group, and $\cdot$ the usual action of the Galilei group onto $\mathbb{R}^4$.

The projection of such a state space onto the second factor is a coadjoint orbit. The corresponding particle belongs to the class whose movement space is this coadjoint orbit.

Each element of a state space is a pair composed of an element of $\mathbb{R}^4$ and a movement of the particle. We interpret this fact by saying that the movement contains the event or "passes across" the event.

With this interpretation, if state space is the orbit of $(r, \alpha) \in \mathbb{R}^4 \times \underline{\tilde{\mathcal{B}}}^*$, the events "contained" in the movement $Ad_L^* \alpha$ are the $\{(Lg) \cdot r\colon g \in \tilde{\mathcal{B}}_\alpha\}$, where $\tilde{\mathcal{B}}_\alpha$ is the isotropy subgroup of $\tilde{\mathcal{B}}$ at α. This set of events is the "general appearance" of the movement in space-time. The movements passing across the event $r' \in \mathbb{R}^4$ are the $\{Ad_L^*\alpha\colon L \cdot r = r'\}$.

The general appearance of a movement in space-time depends, not only on the choice of movement space (i.e., the class of the particle), but also on the choice of state space. More precisely, if $\alpha \in \underline{\tilde{\mathcal{B}}}^*$, $L \in \tilde{\mathcal{B}}$, the movement $Ad_L^*\alpha$ has the same general appearance in space-time when one chooses as classical state space the orbit of (r, α) as if one chooses the orbit of (r', α) if and only if there exist $M \in \tilde{\mathcal{B}}_\alpha$ such that $r' = M \cdot r$, i.e., if the orbits are the same. As an example, in the case of a massive spinless particle, one choice of state space leads to the usual classical movements of a free material point, but other choices lead to other general appearances, which are much less easy to interpret. Similar considerations are to be made with regard to the movements containing a given event.

The elements of the Lie algebra of $\tilde{\mathcal{B}}$ define (linear) functions on $\underline{\tilde{\mathcal{B}}}^*$ and, as a consequence, on each state space. They will be considered as dynamical variables. In particular the functions defined by $\bar{l} \equiv (l^1, l^2, l^3)$, $\bar{p} \equiv (p^1, p^2, p^3)$, E, and $\underline{m}$, (cf. Section 2) will be considered as an abstraction of *angular momentum*, *linear momentum*, *energy*, and *mass*, respectively. These functions together with $\bar{g} \equiv (g^1, g^2, g^3)$ and those given by the canonical coordinates of $\mathbb{R}^4$ give us 15 *canonical dynamical variables*.

The remainder of this section is devoted to proving that this way of looking at classical state space agrees with the usual one, up to diffeomorphism, at least for the classical free massive spinless particle, and to justifying the definition of the canonical dynamical variables we have done. In this section we follow the ideas developed in the relativistic case [8]. Most of them are inspired in by ref. 14.

Configuration space-time for a classical (spinless) free particle with nonzero mass m is interpreted as being an abstract four-dimensional manifold M. Each inertial observer, R is a *global chart* $\phi_R = (x_R^1, x_R^2, x_R^3, t_R)$. Of course M is diffeomorphic to $\mathbb{R}^4$.

We consider a family of inertial observers such that changes of the global charts are given by elements of the connected component of the identity of the Galilei group, i.e., if R and R' are members of the family, there exists (A, b, c, e) in the connected component of the identity of Galilei group such that $\phi_{R'} \circ \phi_R^{-1}$ is given by

$$\begin{aligned} \bar{x}_{R'} &= A\bar{x}_R + bt_R + c \\ t_{R'} &= t_R + e \end{aligned} \tag{3.2}$$

where $\bar{x}_R = {}^t(x_R^1, x_R^2, x_R^3)$.

Charts ϕ_R give rise in the canonical way to charts of TM, $\dot{\phi}_R = (\dot{\bar{x}}_R, \dot{t}_R, \bar{x}_R, t_R) = (\dot{x}_R^1, \dot{x}_R^2, \dot{x}_R^3, \dot{t}_R, x_R^1, x_R^2, x_R^3, t_R)$, where the $\dot{x}_R^i$ and $\dot{t}_R$ are given by $\dot{x}_R^i(v) = v(x_R^i)$, $i = 1, 2, 3$, $\dot{t}_R(v) = v(t_R)$, for all $v \in$ TM. The map $\dot{\phi}_{R'}^{-1} \circ \dot{\phi}_R$ is given by

$$\begin{aligned} \dot{\bar{x}}_{R'} &= A\dot{\bar{x}}_R + b\dot{t}_R \\ \dot{t}_{R'} &= \dot{t}_R \\ \bar{x}_{R'} &= A\bar{x}_R + bt_R + c \\ t_{R'} &= t_R + e \end{aligned}$$

Hence, there exists a submanifold $\mathscr{E}$ of TM given by $\dot{t}_R = 1$ for each inertial observer R.

The restrictions to $\mathscr{E}$ of the $\dot{\phi}_R$ are charts of $\mathscr{E}$, provided that we forget the (constant) $\dot{t}_R$. If we denote by the same letter, maps on TM as its restrictions to $\mathscr{E}$, this chart can be denoted by $\dot{\phi}_R = (\dot{\bar{x}}_R, \bar{x}_R, t_R) = (\dot{x}_R^1, \dot{x}_R^2, \dot{x}_R^3, x_R^1, x_R^2, x_R^3, t_R)$. Thus each inertial observer R associates to an arbitrary point of $\mathscr{E}$, v, seven numbers $\dot{\phi}_R(v)$ which are intepreted as giving velocity, position, and time. Thus points of $\mathscr{E}$ will be called *states* and $\mathscr{E}$ itself, *state space*.

The change of chart $\dot{\phi}_{R'}^{-1} \circ \dot{\phi}_R$ is given by

$$\begin{aligned} \dot{\bar{x}}_{R'} &= A\dot{\bar{x}}_R + b \\ \bar{x}_{R'} &= A\bar{x}_R + bt_R + c \\ t_{R'} &= t_R + e \end{aligned} \tag{3.3}$$

These transformations define an action of the Galilei group on $\mathbb{R}^7$. Let us denote by $(\dot{\bar{x}}, \bar{x}, t)$ the canonical coordinate system in $\mathbb{R}^7$. The differential 2-form Ω_0 on $\mathbb{R}^7$ whose local expression is

$$\Omega_0 = \sum_{i=1}^{3} m d\dot{x}^i \wedge dx^i - dT \wedge dt$$

with

$$T = \frac{1}{2} m \sum_{i=1}^{3} (\dot{x}^i)^2$$

and the vector field whose local expression is

$$X_0 = \dot{x}^i \frac{\partial}{\partial x^i} + \frac{\partial}{\partial t}$$

are left invariant by the Galilei action.

Thus, there exists a well-defined differential 2-form Ω and a vector field X on $\mathscr{E}$ whose local expressions for each inertial observer R are

$$\Omega = \sum_{i=1}^{3} m\, d\dot{x}_R^i \wedge dx_R^i - dT_R \wedge dt_R$$

with

$$T_R = \frac{1}{2} m \sum_{i=1}^{3} (\dot{x}_R^i)^2$$

and

$$X = \dot{x}_R^i \frac{\partial}{\partial x_R^i} + \frac{\partial}{\partial t_R}$$

The motions of the particle under consideration are the trajectories of X in $\mathscr{E}$. We shall prove that $\mathscr{E}$ is canonically diffeomorphic to an orbit of $\tilde{\mathscr{B}}$ in $\mathbb{R}^4 \times \underline{\tilde{\mathscr{B}}}^*$.

As noted, the Galilei action on $\mathbb{R}^7$ preserves Ω_0. We shall see that this action has a "momentum map" from $\mathbb{R}^7$ into the dual of the Galilei Lie algebra [14].

If $Z = (\alpha, \beta, \gamma, \delta) \in (\mathbb{R}^3)^3 \times \mathbb{R}$, we shall denote by $\underline{Z}$ the infinitesimal generator of the action associated to Z, i.e., $\underline{Z}$ is the vector field whose flow is given by the transformations associated to $\{\mathrm{Exp}(-tZ)\}$.

A straightforward computation proves that

$$i_{\underline{Z}}\, \Omega_0 = dF(Z)$$

where

$$F(Z) = \langle \alpha, \bar{x} \times \bar{p} \rangle + \langle \beta, t\bar{p} - m\bar{x} \rangle + \langle \gamma, \bar{p} \rangle - \delta \frac{\langle \bar{p}, \bar{p} \rangle}{2m} + K(Z)$$

K is an arbitrary map from the Lie algebra into $\mathbb{R}$, $\langle \cdot \rangle$ is the ordinary Euclidean product, and $\bar{p} = (p^1, p^2, p^3) = m\dot{\bar{x}}$.

If one choose a linear K, one can define a linear map (the momentum map) μ from $\mathbb{R}^7$ into the dual of the Lie algebra by means of

$$\mu(v) \cdot Z = [F(Z)](v)$$

for all $v \in \mathbb{R}^7$, $Z \in \underline{\mathcal{G}}$.

In terms of the basis $\{(Z_G)_i^*\colon i = 1, \ldots, 10\}$ (cf. Section 2) we have

$$\begin{aligned} \mu = \sum_{i=1}^{3} &[(\bar{x} \times \bar{p})^i (Z_G)_i^* + (t\bar{p} - m\bar{x})^i (Z_G)_{i+3}^* + p^i (Z_G)_{i+6}^*] \\ &- \frac{\langle \bar{p}, \bar{p} \rangle}{2m} (Z_G)_{10}^* + K \end{aligned} \tag{3.4}$$

Let us choose $K = 0$. By standard methods of Kostant–Souriau theory, one can find an action of the Galilei group on the dual of its Lie algebra such that μ becomes equivariant. If one identifies the dual of the Lie algebra with $\mathbb{R}^{10}$ by means of the basis composed of the $(Z_G)_i^*$, the action is given by

$$(A, b, c, e) \cdot \sigma = \begin{pmatrix} A & \hat{b}A & \hat{c}A & 0 \\ 0 & A & eA & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & -{}^t bA & 1 \end{pmatrix} \sigma + m \begin{pmatrix} c \times b \\ eb - c \\ b \\ -b^2/2 \end{pmatrix} \tag{3.5}$$

where ${}^t b$ is the transpose of b, and $b^2 = {}^t bb$.

With this notation one can write

$$\mu = \begin{pmatrix} \bar{x} \times \bar{p} \\ t\bar{p} - m\bar{x} \\ \bar{p} \\ -p^2/2m \end{pmatrix} \tag{3.6}$$

The form of the action (3.5) leads us to consider the Bargmann group

$\mathcal{B}$ or its universal covering group, $\tilde{\mathcal{B}}$ [cf. (2.7) and (2.11)]. Let us consider the action of $\tilde{\mathcal{B}}$ on $\mathbb{R}^7$ defined as in (3.1), for all $L \in \tilde{\mathcal{B}}$, $x \in \mathbb{R}^7$, where now $\cdot$ is the usual action of the Galilei group on $\mathbb{R}^7$.

If we denote by $\underline{Y}$ the infinitesimal generator of the action associated to $Y \in \underline{\tilde{\mathcal{B}}}$, we have

$$i_{\underline{Y}}\, \Omega_0 = i_{\underline{d(\pi \circ Q) \cdot Y}}\, \Omega_0$$

As a consequence of (3.4), one thus sees that

$$\tilde{\mu} = \sum_{i=1}^{3} [(\bar{x} \times \bar{p})^i\, {}^t(dQ) \cdot \overset{*}{Z}_i + (t\bar{p} - m\bar{x})^i\, {}^t(dQ) \cdot \overset{*}{Z}_{i+3} + p^i\, {}^t(dQ) \cdot \overset{*}{Z}_{i+6}]$$
$$- \frac{\langle \bar{p}, \bar{p} \rangle}{2m}\, {}^t(dQ) \cdot \overset{*}{Z}_{10} + M \tag{3.7}$$

is a momentum map for the action (3.1), where M is any fixed element of $\underline{\tilde{\mathcal{B}}}^*$.

If we take $M = -m\, {}^t(dQ) \cdot \overset{*}{Z}_{11}$, we obtain a momentum map which is equivariant for the coadjoint action. In fact, we can identify the dual of the Lie algebra of $\tilde{\mathcal{B}}$ with $\mathbb{R}^{11}$ by means of the basis composed of the ${}^t(dQ) \cdot \overset{*}{Z}_i$, so that the map $\tilde{\mu}$ can be written as

$$\begin{pmatrix} \bar{x} \times \bar{p} \\ t\bar{p} - m\bar{x} \\ \bar{p} \\ -p^2/2m \\ -m \end{pmatrix} \tag{3.8}$$

and the equivariance of $\tilde{\mu}$ with respect to the coadjoint action follows from the equivariance of μ with respect to (3.5).

As a consequence, the image of $\mathbb{R}^7$ by $\tilde{\mu}$ is a coadjoint orbit of $\tilde{\mathcal{B}}$. More precisely, it is the orbit of $(0, \ldots, 0, -m)$ by the coadjoint action. If α is in the coadjoint orbit, its reciprocal image by $\tilde{\mu}$ is the image of an integral curve of X_0, so that $\tilde{\mu}$ establishes a one-to-one correspondence between points of the coadjoint orbit and trajectories of X_0.

Now, let us consider the injective map

$$g\colon \quad (\dot{\bar{r}}, \bar{r}, s) \in \mathbb{R}^7 \to ((\bar{r}, s), \tilde{\mu}(\dot{\bar{r}}, \bar{r}, s)) \in \mathbb{R}^4 \times \underline{\tilde{\mathcal{B}}}^*$$

The map g becomes equivariant when one considers in $\mathbb{R}^4 \times \underline{\tilde{\mathcal{B}}}^*$ the action given by

$$L * (s, \alpha) = (\pi(Q(L)) \cdot s, Ad_L^* \alpha)$$

where the action on $\mathbb{R}^4$ on the right-hand side is the usual one of the Galilei group. In particular, $g(\mathbb{R}^7)$ is an orbit of this action.

For each inertial observer R we define $g_R = g \circ \dot{\phi}_R$ and $\mu_R = \tilde{\mu} \circ \dot{\phi}_R$, so that $g_R = (\phi_R, \mu_R)$.

The map g_R enables us to identify $\mathscr{E}$ with the orbit $g(\mathbb{R}^7)$ of $\tilde{\mathscr{B}}$, but this identification depends on R. In fact, if R and R′ are related by (3.2), then

$$g_{R'} \circ g_R^{-1} = (\phi_{R'} \circ \phi_R^{-1}, Ad^*_{(A,\, b,\, c,\, e,\, a)})$$

where a is arbitrary. Thus, these two ways of identifying state space with the orbit are related by one of the transformations of the action.

The action of $\tilde{\mathscr{B}}$ on $\mathscr{E}$ such that $\dot{\phi}_R$ becomes equivariant obviously preserves Ω, and μ_R is a momentum map for this action.

For all R, the image of μ_R is the coadjoint orbit obtained by projection of $g(\mathbb{R}^7)$ onto $\underline{\mathscr{B}}^*$. Each μ_R maps in a one-to-one way trajectories of X to elements of the coadjoint orbit. Thus the coadjoint orbit will be called the *space of movements*.

Each element of the Lie algebra can be considered as a (linear) function on the dual, and thus on $\mathbb{R}^4 \times \mathscr{B}^*$. With this interpretation we have

$$\begin{aligned} dQ^{-1} \cdot Z^i \circ g_R &= (\bar{x}_R \times \bar{p}_R)^i \\ dQ^{-1} \cdot Z^{i+3} \circ g_R &= (t_R \bar{p}_R - m\bar{x}_R)^i \\ dQ^{-1} \cdot Z^{i+6} \circ g_R &= p_R^i \\ dQ^{-1} \cdot Z^{10} \circ g_R &= -\frac{\langle \bar{p}_R, \bar{p}_R \rangle}{2m} \\ dQ^{-1} \cdot Z^{11} \circ g_R &= -m \end{aligned} \tag{3.9}$$

where $\bar{p}_R = m\dot{x}_R$.

Then

$$\begin{aligned} l^i \circ g_R &= (\bar{x}_R \times \bar{p}_R)^i \\ g^i \circ g_R &= (m\bar{x}_R - t_R \bar{p}_R)^i \\ p^i \circ g_R &= p_R^i \\ E \circ g_R &= \frac{\langle \bar{p}_R, \bar{p}_R \rangle}{2m} \\ \underline{m} \circ g_R &= m \end{aligned} \tag{3.10}$$

which justify our interpretation of the dynamical variables $l^1, \ldots, \underline{m}$.

4. QUANTUM STATES

In this section we define *quantum states* of a particle whose classical state space is a given orbit of $\tilde{\mathscr{B}}$ in $\mathbb{R}^4 \times \underline{\tilde{\mathscr{B}}}^*$ (cf. Section 3).

The idea comes from geometric quantization, in the sense that quantum states correspond to sections of a Hermitian line bundle on a coadjoint orbit (movement space) that are invariant along a certain isotropic foliation. According to the approach developed in the relativistic case [8], the foliation will be fixed by the condition that physics must not depend on the choice of state space corresponding to the given movement space.

For the most part we state the results in terms of the principal circle bundle canonically associated to the Hermitian line bundle. Thus, the starting point will be a Boothby–Wang fibration [4] on the given coadjoint orbit, i.e., a principal bundle with the circle $\mathbb{S}^1$ as structural group, provided with a connection whose curvature form projects onto the canonical symplectic form of the coadjoint orbit.

We also require the total space to be a homogeneous space of $\tilde{\mathcal{B}}$ for an action that is a lift of the coadjoint action (i.e., makes the bundle projection equivariant) and preserves the connection. Under these circumstances the total space is a homogeneous contact manifold.

For the sake of completeness, let us recall some known results concerning the homogeneous contact manifolds under consideration. These results and a study of more general situations can be found in ref. 15 and 5–7. Some of these generalizations also have interest from the point of view of the present paper. In fact, one can consider each covering of a coadjoint orbit as a candidate for movement space, and the geometric construction that follows would remain valid. For the purposes of this paper, it is enough to consider the coadjoint orbits themselves. For fiber bundles, we use the notation of ref. 10.

Let G be a Lie group. A fibration as desired on the coadjoint orbit of $\alpha \in G^*$ exists if and only if there exists a surjective homomorphism, C_α from the isotropy subgroup at α, G_α, onto the unit circle, $\mathbb{S}^1$, whose differential is α. Then α and its coadjoint orbit are said to be *quantizable*. Here, we can consider the differential of a homomorphism onto $\mathbb{S}^1$ as an ordinary 1-form by identifying the Lie algebra of $\mathbb{S}^1$ to $\mathbb{R}$. This identification is defined by the condition that the exponential map becomes $\mathrm{Exp}(a) = e^{2\pi i a}$ for all $a \in \mathbb{R}$. α and its coadjoint orbit are said to be $\mathbb{R}$-*quantizable* if there exists a surjective homomorphism from G_α onto $\mathbb{R}$ whose differential is α. The Lie algebra of $\mathbb{R}$ is identified with $\mathbb{R}$ in such a way that the exponential map becomes the identity. Of course, if α is $\mathbb{R}$-quantizable, it is quantizable. In ref. 7 a slightly more general concept of quantizability is used, but it is unnecesary for the purposes of the present paper.

In what follows we assume that α is quantizable and C_α is an homomorphism from G_α onto the unit circle, whose differential is α. We identify the coadjoint orbit with G/G_α in the canonical way.

We define an action of $\mathbb{S}^1$ on $G/KerC_\alpha$ by means of

$$(g\ Ker\ C_\alpha) * s = gh\ \mathrm{Ker}\ C_\alpha \tag{4.1}$$

where h is any element of G_α such that $C_\alpha(h) = s$. Actually (G/Ker C_α) (G/G_α, $\mathbb{S}^1$) is a principal fiber bundle, the bundle action being the preceding one, and the bundle projection, the canonical map from $G/KerC_\alpha$ onto G/G_α.

The differential 1-form α projects to an invariant contact form ω on $G/KerC_\alpha$.

Let $Z(\omega)$ be the vector field defined by $i_{Z(\omega)}\omega = 1$, $i_{Z(\omega)}\, d\omega = 0$. All the integral curves of $Z(\omega)$ have the same period. If we denote by $T(\omega)$ the period of any integral curve of $Z(\omega)$, then $\omega/T(\omega)$ is a connection form. Since the structural group is Abelian, the curvature form is $d\omega/T(\omega)$. There exist a unique 2-form on G/G_α whose pullback under the bundle map is the curvature form. This form is symplectic and its cohomology class is integral. It will also be called a curvature form. Its reciprocal image under the canonical map of G onto G/G_α is $d\alpha/T(\omega)$. These symplectic manifolds and its covering spaces are Hamiltonian spaces of the group G [11].

The horizontal lift of curves can be described as follows. Given a curve γ in G/G_α, the horizontal lift of γ to $g\ KerC_\alpha$ is

$$\tilde{\gamma}(t) = (\overline{\gamma}(t)\ KerC_\alpha) * \exp\left(-2\pi i \int_{\overline{\gamma}|_{[0,t]}} \alpha\right) \tag{4.2}$$

where $\overline{\gamma}$ is any lifting of γ to G such that $\overline{\gamma}(0) = g$, and the vertical bar means restriction.

Associated to this principal fiber bundle and the canonical action of $\mathbb{S}^1$ on $\mathbb{C}$ one can consider the 1-dimensional vector bundle whose total space is $(G/KerC_\alpha) \times_{\mathbb{S}^1} \mathbb{C}$. This bundle is a complex line bundle, the addition in each fiber is given by

$$[g\ KerC_\alpha, z] + [g'\ KerC_\alpha, z'] = [g\ KerC_\alpha, z + C_\alpha(g^{-1}\, g')z']$$

and the multiplication by complex numbers by

$$a \cdot [g\ KerC_\alpha, z] = [g\ KerC_\alpha, az]$$

This vector bundle becomes Hermitian when one defines in each fiber the Hermitian product

$$\langle [g\ KerC_\alpha, z], [g'\ KerC_\alpha, z'] \rangle = \bar{z} C_\alpha(g^{-1}g')z'$$

It is well known that the sections of the Hermitian line bundle are in one-to-one correspondence with the functions on $G/KerC_\alpha$, f, such that $f((g\ KerC_\alpha) * s) = s^{-1} f(g\ KerC_\alpha)$ for all $s \in \mathbb{S}^1$, $g \in G$. These functions will be called from now on *pseudotensorial functions*. This correspondence is as follows. If f is a pseudotensorial function, the corresponding section sends $m \in G/G_\alpha$ to $[r, f(r)]$, where r is arbitrary in the fiber on m. If σ is a section of the Hermitian line bundle, the corresponding pseudotensorial function f is defined by $\sigma(\pi(r)) = [r, f(r)]$ for all $r \in G/KerC_\alpha$, where π is the bundle projection.

The sections of the Hermitian line bundle are called *prequantum states*. Sometimes we use the same denomination for the corresponding pseudotensorial functions.

Let us return to the case where $G = \tilde{\mathcal{B}}$. As seen in Section 3, there are many candidates for state space for the particle whose movement space is the orbit of α. In fact, for each $(q_0, t_0) \in \mathbb{R}^4$, the orbit of $((q_0, t_0), \alpha) \in \mathbb{R}^4 \times \underline{\mathcal{B}}^*$ is one of them.

If state space is the orbit of $((q_0, t_0), \alpha)$ and $(q, t) \in \mathbb{R}^4$, the movements containing the event (q, t) are the $Ad^*_{(A,b,c,e,a)}\alpha$ such that $(A, b, c, e, a) * (q_0, t_0) = (q, t)$, i.e., the $Ad^*_{(A,b,c,e,a)}\alpha$ such that A, b, and a are arbitrary, $c = q - Aq_0 - bt_0$, and $e = t - t_0$. This set depends on the choice of (q_0, t_0).

We call quantum states the prequantum states which are independent of the preceding choice in the following sense.

Since

$$Ad^*_{(A,b,q-A\cdot q_0-bt_0,t-t_0,a)}\alpha = Ad^*_{(I,0,-A\cdot q_0-bt_0,-t_0,0)} \circ Ad^*_{(A,b,q,t,a)}\alpha$$

we say that a prequantum state, considered as a section, Φ of the Hermitian line bundle, is independent of the choice of (q_0, t_0) if its value on the right-hand side of the preceding equation is independent, up to parallel transport, of the actual value of (q_0, t_0), i.e., if $\phi\ (Ad^*_{(I,0,c,e,0)}\gamma) = \tau(\phi(\gamma))$ for all γ in the orbit and (c, e) in $\mathbb{R}^4$, where τ is the parallel transport along any curve joining γ with $Ad^*_{(I,0,c,e,0)}\gamma$ in the orbit of γ by the subgroup $R_4 = \{(I, 0, q, t, 0)\colon q \in \mathbb{R}^3, t \in \mathbb{R}\}$. An equivalent statement of this condition is that the corresponding pseudotensorial function be constant along the horizontal lift of such a curve; thus we have the following:

Definition 4.1. A *quantum state* is a prequantum state whose corresponding pseudotensorial function is constant along the horizontal lift of any curve whose image is in a orbit of the subgroup R_4.

Let us denote by α_i the components of α in the basis composed by the ${}^t(dQ) \cdot \overset{*}{Z}{}^i$, and $\alpha_{789} = (\alpha_7, \alpha_8, \alpha_9)$.

Lemma 4.2. There exists a unique action of R_4 on $G/KerC_\alpha$ whose orbits are horizontal and such that π becomes equivariant. This action is given by

$$\begin{aligned}(I, 0, q, t, 0) ** ((A, b, c, e, a)\ KerC_\alpha)\\ = ((A, b, q + c, t + e, a)\ KerC_\alpha)\end{aligned}$$

$$* \exp\{-2\pi i[\alpha_{789}\, {}^tA\, (q - tb) + \alpha_{10}\, t + \alpha_{11}\, {}^tb(tb/2 - q)]\} \quad (4.3)$$

for all $q \in \mathbb{R}^3$, $t \in \mathbb{R}$, where $**$ on the left-hand side stands for the new action and $*$ on the right-hand side corresponds to the bundle action.

The proof of this result is very close to that of Lemma 4.1 of ref. 8 and is left to the reader.

This action will be called the *horizontal action*.

Corollary 4.3. The quantum states are the prequantum states that correspond to pseudotensorial functions left invariant by the horizontal action.

Let $(A, b, c, e, a) \in \tilde{\mathcal{B}}$, $q \in \mathbb{R}^3$, $t \in \mathbb{R}$. Let us denote by $(I, 0, q, t, 0)$ $**$ the diffeomorphism associated to $(I, 0, q, t, 0)$ by the horizontal action, and by $(A, b, c, e, a)\cdot$ the diffeomorphism associated to (A, b, c, e, a) by the usual action. Then:

Lemma 4.4. We have

$$\begin{aligned} &((A, b, c, e, a)\cdot) \circ ((I, 0, q, t, 0)\, **) \\ &\quad = ((I, 0, Aq + bt, t, 0)\, **) \circ ((A, b, c, e, a)\cdot) \end{aligned}$$

Proof. Since $(I, 0, 0, 0, a)$ is in the center of $\tilde{\mathcal{B}}$ for all $a \in \mathbb{R}$, its corresponding diffeomorphism by the coadjoint representation is the identity. In particular $(I, 0, 0, 0, a) \in \tilde{\mathcal{B}}_\alpha$ for all $\alpha \in B^*$.

We also have

$$(I, 0, 0, 0, a) = \mathrm{Exp}_{\tilde{\mathcal{B}}}\, (aZ_{11})$$

so that

$$\begin{aligned} C_\alpha((I, 0, 0, 0, a)) &= C_\alpha \circ \mathrm{Exp}_{\tilde{\mathcal{B}}}(aZ_{11}) = \mathrm{Exp}_{\mathbb{S}^1} \circ dC_\alpha(aZ_{11}) \\ &= Exp_{\mathbb{S}^1}\, (a\alpha_{11}) = e^{2\pi i a \alpha_{11}}. \end{aligned} \quad (4.4)$$

For all $(A', b', c', e', a') \in \tilde{\mathcal{B}}$, we thus have

$$\begin{aligned} &(((A, b, c, e, a)\cdot) \circ ((I, 0, q, t, 0)**)((A', b', c', e', a')\, KerC_\alpha) \\ &\quad = ((A, b, c, e, a)(I, 0, q, t, 0)(A', b', c', e', a')\, KerC_\alpha) \\ &\qquad * \exp\{-2\pi i[\alpha_{789}\, {}^tA'(q - tb') + \alpha_{10}t + \alpha_{11}\, {}^tb'(b't/2 - q)]\} \\ &\quad = ((I, 0, Aq + bt, t, 0)(A, b, c, e, a)(I, 0, 0, 0, {}^tbAq + b^2t/2) \\ &\qquad \times (A', b', c', e', a')\, KerC_\alpha) \\ &\qquad * \exp\{-2\pi i[\alpha_{789}\, {}^tA'(q - tb') + \alpha_{10}t + \alpha_{11}\, {}^tb'(b't/2 - q)]\} \end{aligned}$$

$$= ((I, 0, Aq + bt, t, 0)(A, b, c, e, a)(A', b', c', e', a')$$
$$\times (I, 0, 0, 0, {}^t bAq + b^2 t/2)\ KerC_\alpha)$$
$$* \exp\{-2\pi i[\alpha_{789}\ {}^t A'(q - tb') + \alpha_{10} t + \alpha_{11}\ {}^t b'(b't/2 - q)]\} \qquad (4.5)$$

As a consequence of (4.4), the last of (4.5) coincides with

$$((I, 0, Aq + bt, t, 0)(A, b, c, e, a)(A', b', c', e', a')\ KerC_\alpha)$$
$$* \exp\{-2\pi i[\alpha_{789}\ {}^t A'(q - tb') + \alpha_{10} t + \alpha_{11}\ {}^t b'(b't/2 - q)]$$
$$- \alpha_{11}({}^t bAq + b^2 t/2)\} \qquad (4.6)$$

and a straightforward computation proves that this expression coincides with

$$(((I, 0, Aq + bt, t, 0)**) \circ ((A, b, c, et, a)\cdot))((A', b', c', e', a')\ KerC_\alpha) \quad \blacksquare$$

It follows that:

Corollary 4.5. The usual action transforms horizontal orbits into horizontal orbits.

The usual action thus defines a transitive action in the set of horizontal orbits. The isotropy subgroup at the horizontal orbit of $KerC_\alpha$ is composed of the (A, b, c, e, a) such that there exist $q \in \mathbb{R}^3$, $t \in \mathbb{R}$, such that

$$\begin{aligned}(A, b, c, e, a)\ KerC_\alpha &= (I, 0, q, t, 0) ** KerC_\alpha \\ &= (I, 0, q, t, 0)\ KerC_\alpha * e^{-2\pi i(\alpha_{789} q + \alpha_{10} t)} \\ &= (I, 0, q, t, 0) g\ KerC_\alpha\end{aligned}$$

where g is an element of $\tilde{\mathcal{B}}_\alpha$ such that

$$C_\alpha(g) = e^{-2\pi i(\alpha_{789} q + \alpha_{10} t)}$$

As a consequence we have.

Lemma 4.6. The isotropy subgroup at the horizontal orbit of $KerC_\alpha$ is composed of the $(I, 0, q, t, 0)g$, $q \in \mathbb{R}^3$, $t \in \mathbb{R}$, $g \in \tilde{\mathcal{B}}_\alpha$, such that

$$C_\alpha(g)\ e^{2\pi i(\alpha_{789} q + \alpha_{10} t)} = 1$$

In what follows we characterize this subgroup as the kernel of a certain homomorphism.

Let $(\tilde{\mathcal{B}}_\alpha)_0$ be the image of $\tilde{\mathcal{B}}_\alpha$ under the homomorphism π_1: $(A, b, c, e, a) \in \tilde{\mathcal{B}} \to (A, b) \in \tilde{\mathcal{B}}_0$.

Lemma 4.7. The map from $(\tilde{\mathcal{B}}_\alpha)_0$ into $\mathbb{S}^1$, $(C_\alpha)_0$, given by

$$(C_\alpha)_0(A, b) = C_\alpha(A, b, c, e, a)\, e^{-2\pi i(\alpha_{789}c+\alpha_{10}e+\alpha_{11}a)}$$

for all $(A, b, c, e, a) \in \mathcal{B}_\alpha$ is a well-defined homomorphism.

Proof. The kernel of $\pi_1|_{\tilde{\mathcal{B}}_\alpha}$ is obviously contained in $\{(I, 0)\} \times \mathbb{R}^5$. From equation (2.11) it follows that the kernel is composed of the $(I, 0, c, e, a) \in \tilde{\mathcal{B}}$ such that

$$\hat{c}\begin{pmatrix}\alpha_7\\ \alpha_8\\ \alpha_9\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \qquad -e\begin{pmatrix}\alpha_7\\ \alpha_8\\ \alpha_9\end{pmatrix} + c\alpha_{11} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Since these are linear equations, the kernel is arcwise connected.

If $(A, b, c, e, a), (A, b, c', e', a') \in \mathcal{B}_\alpha$, we have $(A, b, c, e, a)(A, b, c', e', a')^{-1} = (I, 0, c - c', e - e', a - a') \in Ker(\pi_1|_{\tilde{\mathcal{B}}_\alpha})$. Then we define $\gamma(t) = (I, 0, t(c - c'), t(e - e'), t(a - a'))$, which is a curve in $Ker(\pi_1|_{\tilde{\mathcal{B}}_\alpha})$, and we have [7]

$$\begin{aligned} &C_\alpha((A, b, c, e, a)(A, b, c', e', a')^{-1}) \\ &\quad = \exp\left(2\pi i \int_\gamma \alpha\right) \\ &\quad = \exp\{2\pi i[\alpha_{789}(c - c') + \alpha_{10}(e - e') + \alpha_{11}(a - a')]\} \end{aligned}$$

Then we have

$$\begin{aligned} &C_\alpha(A, b, c, e, a)\, e^{-2\pi i(\alpha_{789}c+\alpha_{10}e+\alpha_{11}a)} \\ &\quad = C_\alpha(A, b, c', e', a')\, e^{-2\pi i(\alpha_{789}c'+\alpha_{10}e'+\alpha_{11}a')} \end{aligned}$$

This proves that $(C_\alpha)_0$ is well defined. One can convince oneself of the fact that $(C_\alpha)_0$ is a homomorphism by direct computation. ■

We also define $\tilde{C}_\alpha: (\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5 \to \mathbb{S}^1$ by means of

$$\tilde{C}_\alpha(A, b, c, e, a) = ((C_\alpha)_0(A, b))\, e^{2\pi i(\alpha_{789}c+\alpha_{10}e+\alpha_{11}a)}$$

A straightforward computation proves that $\tilde{C}_\alpha$ is a homomorphism. It is obviously an extension of C_α.

Lemma 4.8. The isotropy subgroup at the horizontal orbit of $KerC_\alpha$ is $Ker\tilde{C}_\alpha$.

Proof. As a consequence of Lemma 4.6, any element of the isotropy subgroup has the form $(I, 0, q, t, 0)g$, where $q \in \mathbb{R}^3$, $t \in \mathbb{R}$, $g \in \tilde{\mathcal{B}}_\alpha$, so that it is in $(\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5$, and moreover

$$1 = C_\alpha(g)\, e^{2\pi i(\alpha_{789}q+\alpha_{10}t)} = \tilde{C}_\alpha(g)\tilde{C}_\alpha(I, 0, q, t, 0) = \tilde{C}_\alpha((I, 0, q, t, 0)g)$$

It follows that the isotropy subgroup is in $Ker\tilde{C}_\alpha$.

On the other hand, let $(A, b, c, e, a) \in \mathcal{B}_\alpha$, $c_0 \in \mathbb{R}^3$, e_0, $a_0 \in \mathbb{R}$, be such that $(A, b, c_0, e_0, a_0) \in Ker\tilde{C}_\alpha$. Then $(A, b, c, e, a_0) = (I, 0, 0, 0, a_0 - a)(A, b, c, e, a)$, so that (A, b, c, e, a_0) is in $\mathcal{B}_\alpha$, and we have

$$(A, b, c_0, e_0, a_0) = (I, 0, c_0 - c, e_0 - e, 0)(A, b, c, e, a_0)$$

and

$$\begin{aligned} &C_\alpha(A, b, c, e, a_0)\, e^{2\pi i(\alpha_{789}(c_0-c)+\alpha_{10}(e_0-e))} \\ &\quad = \tilde{C}_\alpha(A, b, c, e, a_0)\tilde{C}_\alpha(I, 0, c_0 - c, e_0 - e, 0) \\ &\quad = \tilde{C}_\alpha(I, 0, c_0 - c, e_0 - e, 0)\tilde{C}_\alpha(A, b, c, e, a_0) \\ &\quad = \tilde{C}_\alpha((I, 0, c_0 - c, e_0 - e, 0)(A, b, c, e, a_0)) = 1 \end{aligned}$$

It follows that (A, b, c_0, e_0, a_0) is in the isotropy subgroup. ■

The space of orbits is thus canonically bijective to $\tilde{\mathcal{B}}/Ker\tilde{C}_\alpha$.

In the remainder of this section we assume that α is such that $(\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5$ is closed in $\tilde{\mathcal{B}}$ and has a finite number of connected components and that $\tilde{C}_\alpha$ is continuous. These facts actually hold for all quantizable α, but will not be proved here. A proof can be obtained by direct verification, using a classification of the coadjoint orbits of $\tilde{\mathcal{B}}$ that Jaime Hoyos has found and will be published elsewhere.

Thus, $Ker\tilde{C}_\alpha$ is a closed subgroup of $\tilde{\mathcal{B}}$, so that $\tilde{\mathcal{B}}/Ker\tilde{C}_\alpha$ has the canonical structure of a differentiable manifold. We consider the space of orbits of the horizontal action provided with the topology and differentiable structure which makes the canonical bijection onto $\tilde{\mathcal{B}}/Ker\tilde{C}_\alpha$ a diffeomorphism.

Since $\mathcal{B}_\alpha \subset (\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5$ and $KerC_\alpha \subset Ker\tilde{C}_\alpha$, we have canonical maps such that the diagram in Fig. 1 is commutative.

The homomorphism $\tilde{C}_\alpha$ gives us an isomorphism of $(\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5/Ker\tilde{C}_\alpha$ onto $\mathbb{S}^1$. If we identify these groups by means of that homomorphism, we obtain a principal fiber bundle $\tilde{\mathcal{B}}/Ker\tilde{C}_\alpha\,(\tilde{\mathcal{B}}/((\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5), \mathbb{S}^1)$. The horizontal arrows in Fig. 1 give us a homomorphism of principal circle bundles.

$$\begin{tikzcd} \tilde{\mathcal{B}}/KerC_\alpha \arrow[r] \arrow[d] & \tilde{\mathcal{B}}/Ker\tilde{C}_\alpha \arrow[d] \\ \tilde{\mathcal{B}}/\tilde{\mathcal{B}}_\alpha \arrow[r] & \tilde{\mathcal{B}}/((\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5) \end{tikzcd}$$

Fig. 1. Fiber bundles for geometric quantum states.

The upper horizontal arrow corresponds to the map that sends each element of $\tilde{\mathcal{B}}/KerC_\alpha$ to its horizontal orbit. Since quantum states correspond to pseudotensorial functions left invariant by the horizontal action, they will be identified with unrestricted pseudotensorial functions on $\tilde{\mathcal{B}}/Ker\tilde{C}_\alpha$.

5. WAVE FUNCTIONS

Let H and K be closed subgroups of a Lie group G. Let us assume that there exists an action of G on the left on H, $(g, h) \in G \times H \to g * h \in H$, such that $(g * h)^{-1}\, gh \in K$ for all $g \in G$, $h \in H$.

We define

$$\nu: \quad (g, h) \in G \times H \to (g * h)^{-1} gh \in K$$

For an arbitrary closed subgroup L of G a straightforward computation proves that

$$\Phi: \quad (g, (h, kK \cap L)) \in G \times \left(H \times \frac{K}{K \cap L}\right) \to (g * h, \nu(g, h)kK \cap L) \in H \times \frac{K}{K \cap L} \tag{5.1}$$

is an action on the left.

Example 1. Let H and K be Lie groups and s a homomorphism from K into the group of the automorphisms of H. In the semidirect product $K \times_s H$ the subgroups $\{e\} \times H$ and $K \times \{e\}$ are canonically isomorphic to H and K, respectively, where e represents the identity element of K and H, respectively.

We define an action of $K \times_s H$ on $\{e\} \times H$ by means of

$$(k, h) * (e, h') = (e, hs_k(h'))$$

where s_k stands for $s(k)$, and we see that the conditions hold with $\nu\,((k, h), (e, h')) = (k, e)$.

A particular case is the Poincaré group. The geometrical construction we made in this section in general has been made in the case of the Poincaré group in ref. 8.

Example 2. The group $\tilde{\mathcal{B}}$ is a semidirect product (cf. the end of Section 2), so that we can do the preceding construction, according to Example 1, with $H = \mathbb{R}^5$ and $K = \tilde{\mathcal{B}}_0$.

Now we proceed otherwise. Let $H = R_4$ and $K = \{(A, b, 0, 0, a)\colon A \in SU\,(2),\ b \in \mathbb{R}^3,\ a \in \mathbb{R}\}$. We define an action of $\tilde{\mathcal{B}}$ on H by means of

$$(A, b, c, e, a) * (I, 0, x, t, 0) = (I, 0, Ax + bt + c, e + t, 0)$$

where Ax stands for $q(A)x$, as said at the end of Section 2, and we see in this case that

$$\nu((A, b, c, e, a), (I, 0, x, t, 0)) = (A, b, 0, 0, a + {}^t b(Ax + tb/2))$$

The corresponding action of $\tilde{\mathcal{B}}$ on $H \times (K/(K \cap L))$ is given by

$$\begin{aligned}
&\Phi((A, b, c, e, a), ((I, 0, x, t, 0), (A', b', 0, 0, a')K \cap L)) \\
&\quad = ((I, 0, Ax + bt + c, t + e, 0), (AA', Ab' + b, 0, 0, a + a' \\
&\qquad + {}^t b(Ax + tb/2))K \cap L)
\end{aligned} \tag{5.2}$$

Returning to the general case, if we denote by e the identity element of G, we have:

Proposition 5.1. If $K \cap H = \{e\}$, the actions of G on H and on $H \times (K/(K \cap L))$ are both transitive. The isotropy subgroup of G at e is K for the action on H. The isotropy subgroup of G at $(e, K \cap L)$ is $K \cap L$ for the action on $H \times (K/(K \cap L))$.

Proof. If $h, h' \in H$, we have $\nu(h, h') = (h * h')^{-1}hh'$, so that $\nu(h, h') \in K \cap H = \{e\}$. Then $h * h' = hh'$ and, as a particular consequence, the action on H is transitive.

Let $Isotr(e)$ be the isotropy subgroup of G at e for the action on H and $Isotr(e, K \cap L)$ be the isotropy subgroup of G at $(e, K \cap L)$ for the action on $H \times (K/(K \cap L))$.

If $g \in Isotr(e)$, we have $\nu(g, e) = (g * e)^{-1}ge = g$, so that $g \in K$. If $k \in K$, we have $\nu(k, e) = (k * e)^{-1}k$, so that $k * e \in K \cap H = \{e\}$ and it follows that $k \in Isotr(e)$. As a consequence $Isotr(e) = K$.

Now, let $h \in H$, $k \in K$. We have

$$\begin{aligned}
\Phi(hk, (e, K \cap L)) &= ((hk) * e, \nu(hk, e)K \cap L) \\
&= ((hk) * e, ((hk) * e)^{-1}hkK \cap L) \\
&= (h * e, (h * e)^{-1}hkK \cap L) = (h, k\,K \cap L)
\end{aligned}$$

which implies that the action in $H \times (K/(K \cap L))$ is transitive.

On the other hand, $g \in Isotr(e, K \cap L)$ if and only if $g * e = e$ and $\nu(g, e) \in K \cap L$, i.e., if and only if $g \in K \cap L$. ■

In the case $K \cap H = \{e\}$, we thus obtain an equivariant diffeomorphism φ_0^{-1} from $G/(K \cap L)$ onto $H \times K/(K \cap L)$ given by

$$\begin{aligned}
\varphi_0^{-1}(gK \cap L) &= g * (e, K \cap L) \\
&= (g * e, \nu(g, e)\, K \cap L) \\
&= (g * e, (g * e)^{-1}gK \cap L)
\end{aligned}$$

whose inverse is obviously given by

$$\varphi_0\colon \quad (h, kK \cap L) \in H \times \frac{K}{K \cap L} \to hkK \cap L \in \frac{G}{K \cap L} \tag{5.3}$$

The composition of that map with the canonical one leads us to the surjective submersion given by

$$\varphi\colon \quad (h, kK \cap L) \in H \times \frac{K}{K \cap L} \to hkL \in \frac{G}{L} \tag{5.4}$$

which also is equivariant.

Remark 5.2. For each h fixed in H, the restriction of φ to the subset $\{h\} \times (K/(K \cap L))$ is easily seen to be injective.

In the remainder of this section we only consider the four cases arising from Example 2 when L is each one of the denominators in the homogeneous spaces that appear in Fig.1. The maps given by (5.4) in the four cases under consideration are denoted by (5)–(8). The maps appearing in the diagram of Fig. 1 are denoted by (1)–(4) as indicated in Fig. 2.

Since the group law in K is

$$(A, b, 0, 0, a) \perp (A', b', 0, 0, a') = (AA', Ab' + b, 0, 0, a + a')$$

we see that K is isomorphic to the direct product of $\widetilde{\mathcal{B}}_0$ by $\mathbb{R}$. Thus, we denote K by $\widetilde{\mathcal{B}}_0 \times \mathbb{R}$ in what follows. In the same way, the subgroup of K

$$\begin{tikzcd}
R_4 \times \frac{\widetilde{\mathcal{B}}_0 \times \mathbb{R}}{Ker C_\alpha \cap (\widetilde{\mathcal{B}}_0 \times \mathbb{R})} \arrow[rrr, "(5)"] \arrow[dr, "(11)"] \arrow[ddd, "(9)"'] & & & \frac{\widetilde{\mathcal{B}}}{Ker C_\alpha} \arrow[dl, "(1)"'] \arrow[ddd, "(3)"] \\
& R_4 \times \frac{\widetilde{\mathcal{B}}_0 \times \mathbb{R}}{Ker \widetilde{C}_\alpha \cap (\widetilde{\mathcal{B}}_0 \times \mathbb{R})} \arrow[r, "(6)"] \arrow[d, "(12)"'] & \frac{\widetilde{\mathcal{B}}}{Ker \widetilde{C}_\alpha} \arrow[d, "(2)"] & \\
& R_4 \times \frac{\widetilde{\mathcal{B}}_0 \times \mathbb{R}}{\left((\widetilde{\mathcal{B}}_\alpha)_{0 \times_r \mathbb{R}^5}\right) \cap (\widetilde{\mathcal{B}}_0 \times \mathbb{R})} \arrow[r, "(7)"] & \frac{\widetilde{\mathcal{B}}}{(\widetilde{\mathcal{B}}_\alpha)_{0 \times_r \mathbb{R}^5}} & \\
R_4 \times \frac{\widetilde{\mathcal{B}}_0 \times \mathbb{R}}{\widetilde{\mathcal{B}}_\alpha \cap (\widetilde{\mathcal{B}}_0 \times \mathbb{R})} \arrow[ur, "(10)"] \arrow[rrr, "(8)"] & & & \frac{\widetilde{\mathcal{B}}}{\widetilde{\mathcal{B}}_\alpha} \arrow[ul, "(4)"']
\end{tikzcd}$$

Fig. 2. Fiber bundles for quantum states. First diagram.

composed of the $(A, b, 0, 0, a)$ such that $(A, b) \in ((\tilde{\mathcal{B}}_\alpha)_0$ is denoted by $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$.

The spaces where the maps (5)–(8) are defined are homogeneous spaces and there exist canonical projections between them whose product by the identity map of R_4 will be denoted by (9)–(12). All these geometrical objects are related by the commutative diagram appearing in Fig. 2.

We obviously have $((\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5) \cap (\tilde{\mathcal{B}}_0 \times \mathbb{R}) = (\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$. On the other hand, the maps

$$(A, b)(\tilde{\mathcal{B}}_\alpha)_0 \in \frac{\tilde{\mathcal{B}}_0}{(\tilde{\mathcal{B}}_\alpha)_0} \to (A, b, 0, 0, 0)(\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5 \in \frac{\tilde{\mathcal{B}}}{(\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5}$$

and

$$(A, b)(\tilde{\mathcal{B}}_\alpha)_0 \in \frac{\tilde{\mathcal{B}}_0}{(\tilde{\mathcal{B}}_\alpha)_0} \to (A, b, 0, 0, 0)(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R} \in \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}}$$

are easily seen to be diffeomorphisms. When these spaces are identified by means of these diffeomorphisms, the map (7) becomes the projection onto the second factor.

Let us denote by C'_α (resp. $\tilde{C}'_\alpha$) the restrictions of C_α (resp. $\tilde{C}_\alpha$) to $\tilde{\mathcal{B}}_\alpha \cap (\tilde{\mathcal{B}}_0 \times \mathbb{R})$ (resp. $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$). Then the commutative diagram in Fig. 2 becomes the commutative diagram in Fig. 3, where now (6) is the map sending $((I, 0, x, t, 0), (A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha))$ to $(A, b, x, t, a)\ Ker\tilde{C}_\alpha$.

$$\begin{tikzcd}
R_4 \times \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{KerC'_\alpha} \arrow[rrr, "(5)"] \arrow[ddd, "(9)"'] \arrow[dr, "(11)"] & & & \frac{\tilde{\mathcal{B}}}{KerC_\alpha} \arrow[ddd, "(3)"] \arrow[dl, "(1)"'] \\
& R_4 \times \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{Ker\tilde{C}'_\alpha} \arrow[r, "(6)"] \arrow[d, "(12)"'] & \frac{\tilde{\mathcal{B}}}{Ker\tilde{C}_\alpha} \arrow[d, "(2)"] & \\
& R_4 \times \frac{\tilde{\mathcal{B}}_0}{(\tilde{\mathcal{B}}_\alpha)_0} \arrow[r, "(7)"] & \frac{\tilde{\mathcal{B}}_0}{(\tilde{\mathcal{B}}_\alpha)_0} & \\
R_4 \times \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{\tilde{\mathcal{B}}_\alpha \cap (\tilde{\mathcal{B}}_0 \times \mathbb{R})} \arrow[rrr, "(8)"] \arrow[ur, "(10)"] & & & \frac{\tilde{\mathcal{B}}}{\tilde{\mathcal{B}}_\alpha} \arrow[ul, "(4)"']
\end{tikzcd}$$

Fig. 3. Fiber bundles for quantum states.

Let us comment on the physical significance of the botton row in the diagram. The homogeneous space $\tilde{\mathcal{B}}/\tilde{\mathcal{B}}_\alpha$ is identified in the canonical way to the coadjoint orbit of α, i.e., to movement space. Let us assume that the origin in the space of events $0 \in \mathbb{R}^4$ is one of the events contained in the movement α, i.e., that state space is the orbit of $(0, \alpha)$ in $\mathbb{R}^4 \times \underline{\tilde{\mathcal{B}}}^*$.

The isotropy subgroup at $(0, \alpha)$ is $(\tilde{\mathcal{B}}_0 \times \mathbb{R}) \cap \tilde{\mathcal{B}}_\alpha$, so that state space can be identified with $\hat{\tilde{\mathcal{B}}}/(\tilde{\mathcal{B}}_0 \times \mathbb{R}) \cap \tilde{\mathcal{B}}_\alpha$. But φ_0^{-1} [cf. equation (5.3)] enables us to identify this space with

$$R_4 \times \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{\tilde{\mathcal{B}}_\alpha \cap (\tilde{\mathcal{B}}_0 \times \mathbb{R})}$$

When this identification is made, the canonical map from states space onto movement space becomes the map (8). Thus the map (7) must be considered as being a canonical map from "equivalence classes" of states onto "equivalence classes" of movements.

As a consequence of Remark 5.2, the restriction of (8) to

$$\{(I, 0, x, t, 0)\} \times \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{\tilde{\mathcal{B}}_\alpha \cap (\tilde{\mathcal{B}}_0 \times \mathbb{R})}$$

establishes a bijection from this set onto $\{(A, b, x, t, a)\tilde{\mathcal{B}}_\alpha: A \in SU(2), b \in \mathbb{R}^3, a \in \mathbb{R}\}$, which is the set of movements containing (x, t). When one identifies, as above, state space with $R_4 \times ((\mathcal{B}_0 \times \mathbb{R})/(\mathcal{B}_0 \times \mathbb{R} \cap \mathcal{B}_\alpha))$, the set $\{(I, 0, x, t, 0)\} \times ((\mathcal{B}_0 \times \mathbb{R})/(\mathcal{B}_0 \times \mathbb{R} \cap \tilde{\mathcal{B}}_\alpha))$ is thus identified with the set of possible states in the event (x, t), each of these states corresponding to a movement passing through (x, t). In particular, the same differentiable manifold $(\mathcal{B}_0 \times \mathbb{R})/(\mathcal{B}_0 \times \mathbb{R} \cap \tilde{\mathcal{B}}_\alpha)$ parametrizes the set of movements passing though an arbitrary event. Notice that when two different events are given, $(x, t), (x', t') \in \mathbb{R}^4$, an element of $(\mathcal{B}_0 \times \mathbb{R})/(\mathcal{B}_0 \times \mathbb{R} \cap \tilde{\mathcal{B}}_\alpha)$ represents a movement passing through (x, t) and a movement passing through (x', t'), but these movements are in general different.

With the identifications we have done, (2) is the map that sends $(A, b, c, e, a)\ Ker\tilde{C}_\alpha \in \tilde{\mathcal{B}}/Ker\tilde{C}_\alpha$ to $(A, b)(\tilde{\mathcal{B}}_\alpha)_0 \in \tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$.

The map (12) is the product of the identity of R_4 by the map defined by sending $(A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha \in \tilde{\mathcal{B}}/Ker\tilde{C}'_\alpha$ to $(A, b)(\tilde{\mathcal{B}}_\alpha)_0 \in \tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$, and is the bundle projection of a principal fiber bundle whose structural group is $((\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R})/Ker\tilde{C}'_\alpha$.

The homomorphism $\tilde{C}'_\alpha$ establishes a canonical isomorphism $\tilde{C}'_\alpha$ of $((\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R})/Ker\tilde{C}'_\alpha$ onto the subgroup $S = \tilde{C}'_\alpha((\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R})$ of $\mathbb{S}^1$.

Lemma 5.3. The subgroup S is closed in $\mathbb{S}^1$ and $\tilde{C}'_\alpha$ is a Lie group isomorphism.

Proof. If the differential of $\tilde{C}'_\alpha$ is not zero, the subgroup S contains a neighborhood of the identity element of $\mathbb{S}^1$, so that it is all of $\mathbb{S}^1$. Then $\tilde{C}'_\alpha$ is easily seen to be a diffeomorphism.

If the differential vanishes, $\tilde{C}'_\alpha$ is constant on each connected component so that, as a consequence of the fact that the number of connected components of $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$ is finite, $\tilde{C}'_\alpha((\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R})$ is finite. The map $\tilde{C}'_\alpha$ is in this case an isomorphism of finite Lie groups. ∎

We identify the Lie group $((\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R})/Ker\tilde{C}'_\alpha$ with S by means of $\tilde{C}'_\alpha$.

The principal fiber bundle

$$\left(R_4 \times \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{Ker\tilde{C}'_\alpha}\right)\left(R_4 \times \frac{\tilde{\mathcal{B}}_0}{(\tilde{\mathcal{B}}_\alpha)_0}, S\right)$$

whose bundle projection is (12), will be denoted by Π_0. The principal fiber bundle

$$\left(\frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{Ker\tilde{C}'_\alpha}\right)\left(\frac{\tilde{\mathcal{B}}_0}{(\tilde{\mathcal{B}}_\alpha)_0}, S\right)$$

will be denoted by Π and its bundle projection by π.

Now we are interested in a characterization of the pullback by (6) of the quantum states.

As a consequence of Equation (2.11), we see that the map

$$(U, b) \in \tilde{\mathcal{B}}_0 \to \begin{pmatrix} q(U) & 0 & -b \\ -{}^t bq(U) & 1 & b^2/2 \\ 0 & 0 & 1 \end{pmatrix} \in GL(5, \mathbb{R}) \tag{5.5}$$

is a representation and also that, if $(U_0, b_0) \in (\tilde{\mathcal{B}}_\alpha)_0$, we must have

$$\begin{pmatrix} q(U_0) & 0 & -b_0 \\ -{}^t b_0 q(U) & 1 & b_0^2/2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix} = \begin{pmatrix} \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix} \tag{5.6}$$

Then, the following map is well defined:

$$(U, b)(\tilde{\mathcal{B}}_\alpha)_0 \in \tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0 \to \begin{pmatrix} q(U) & 0 & -b \\ -{}^t bq(U) & 1 & b^2/2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix} \in \mathbb{R}^5 \tag{5.7}$$

We denote by P the function defined on $\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$ with values in $\mathbb{R}^4$ given by the first four components of the map (5.7).

The function defined on the coadjoint orbit whose components are $(p^1, p^2, p^3, -E)$ (cf. Section 3) takes in $Ad^*_{(U,b,c,e,a)}\alpha$ the value $P((U, b)(\tilde{\mathcal{B}})_\alpha)_0)$. Then, if we identify the coadjoint orbit with $\tilde{\mathcal{B}}/\mathcal{B}_\alpha$, we see that the function ${}^t(p^1, p^2, p^3, -E)$ is projectable in $\tilde{\mathcal{B}}/(\tilde{\mathcal{B}}_\alpha)_0$ under (4) and its projection is P.

Theorem 5.4. The pullback by (6) maps in a one-to-one way the set of quantum states onto the set composed of the functions on $R_4 \times (\tilde{\mathcal{B}}_0 \times \mathbb{R})/Ker\tilde{C}'_\alpha$ having the form

$$W_f((I, 0, x, t, 0), (A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha) = f((A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha)\ e^{-2\pi i\langle (x,t),\ P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle} \tag{5.8}$$

where f is a pseudotensorial function of the principal fiber bundle Π, and $\langle\cdot, \cdot\rangle$ stands for the Euclidean product of $\mathbb{R}^4$.

Proof. We denote by χ_0 the complex-valued function on $R_4 \times (\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0)$ given by

$$\chi_0((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0) = e^{2\pi i\langle (x,t),\ P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}$$

The map $\chi \equiv \chi_0 \circ (12)$ is such that

$$\chi((I, 0, x, t, 0), (A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha) = e^{2\pi i\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}$$

Let F be a quantum state (i.e., a pseudotensorial function on $\tilde{\mathcal{B}}/Ker\tilde{C}_\alpha$) and $\varphi = F \circ (6)$. We have

$$\begin{aligned}
&(\varphi\chi)((I, 0, x, t, 0), (A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha)\\
&\quad = e^{2\pi i\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle} F(A, b, x, t, a)\ Ker\tilde{C}_\alpha)\\
&\quad = e^{2\pi i\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}\\
&\qquad \times F((A, b, 0, 0, a)(I, 0, {}^tA(x - bt), t, (b^2/2)t - {}^tbx)\ Ker\tilde{C}_\alpha)\\
&\quad = e^{2\pi i\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}\\
&\qquad \times [\tilde{C}_\alpha(I, 0, {}^tA(x - bt), t, (b^2/2)t - {}^tbx)]^{-1}\ F((A, b, 0, 0, a)\ Ker\tilde{C}_\alpha)\\
&\quad = e^{2\pi i\{\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle - (\alpha_{789}{}^tA(x-bt) + \alpha_{10}t + \alpha_{11}((b^2/2)t - {}^tbx))\}}\\
&\qquad \times F((A, b, 0, 0, a)\ Ker\tilde{C}_\alpha)\\
&\quad = F((A, b, 0, 0, a)\ Ker\tilde{C}_\alpha)
\end{aligned}$$

Thus φ has the form W_f where f is the pullback of F by the canonical projection of $(\tilde{\mathcal{B}}_0 \times \mathbb{R})/Ker\tilde{C}'_\alpha$ onto $\mathcal{B}/Ker\tilde{C}_\alpha$.

Now, let f be a pseudotensorial function of Π. We prove in what follows that W_f is projectable by (6) and the projection is pseudotensorial. To prove all that, it suffices to see that the relation

$$\begin{aligned}(I, 0, x, t, 0)(A, b, 0, 0, a)\ Ker\tilde{C}_\alpha \\ = (I, 0, x', t', 0)(A', b', 0, 0, a')\ Ker\tilde{C}_\alpha * s\end{aligned} \tag{5.9}$$

imply

$$\begin{aligned}W_f((I, 0, x, t, 0)(A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha) \\ = \bar{s}W_f((I, 0, x', t', 0)(A', b', 0, 0, a')\ Ker\tilde{C}'_\alpha)\end{aligned} \tag{5.10}$$

for all $s \in \mathbb{S}^1$, $x, b \in \mathbb{R}^3$, $a \in \mathbb{R}$, $A \in SU(2)$.

If equation (5.9) holds, there exist $(A_0, b_0, c_0, e_0, a_0) \in (\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5$ such that $\tilde{C}_\alpha(A_0, b_0, c_0, e_0, a_0) = s$ and $(A, b, x, t, a) = (A', b', x', t', a')(A_0, b_0, c_0, e_0, a_0)$. Then $(A, b) = (A', b')(A_0, b_0)$, $x = A'c_0 + b'e_0 + x'$, $t = t' + e_0$, $a = a' + a_0 + {}^t b'A'c_0 + b'^2 e_0/2$ and

$$\begin{aligned}&W_f((I, 0, x, t, 0)(A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha) \\ &= f((A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha)\ e^{-2\pi i\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle} \\ &= f(((A', b')(A_0, b_0), 0, 0, 0)\ Ker\tilde{C}'_\alpha)\ e^{-2\pi i \alpha_{11} a}\ e^{-2\pi i\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle} \\ &= f((A', b', 0, 0, 0)\ Ker\tilde{C}'_\alpha)\overline{(C_\alpha)_0(A_0, b_0)\ e^{2\pi i(\alpha_{11}a + \langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle)}} \\ &= f((A', b', 0, 0, a')\ Ker\tilde{C}'_\alpha)\ \overline{(C_\alpha)_0(A_0, b_0)\ e^{2\pi i(\alpha_{11}(a-a') + \langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle)}}\end{aligned}$$

but a straightforward computation proves that

$$\begin{aligned}\langle (x, t), P((A, b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle &= \alpha_{789}c_0 + \alpha_{10}e_0 + \alpha_{11}a_0 \\ &\quad + \langle (x', t'), P((A', b')(\tilde{\mathcal{B}}_\alpha)_0)\rangle + \alpha_{11}(a' - a)\end{aligned}$$

and (5.10) follows. ■

The map defined by sending each quantum state F to the f such that $F \circ (6) = W_f$ is an isomorphism [ps] of the complex vector space of quantum states onto the complex vector space PS of pseudotensorial functions of Π.

The preceding theorem enables us to interpret quantum states as pseudotensorial functions of Π_0, or sections of the associated Hermitian line bundle. The interesting fact in this description is that these sections depend on two separate variables, one of them describing an event and the other a class of movements containing that event.

Our next step toward wave functions is to establish an isomorphism v from the complex vector space composed of these sections of an, in general, nontrivial line bundle onto a complex vector space *PW* of functions on the base space with values in a finite-dimensional complex vector space. The elements of *PW* will be called *prewave functions*. This will enable us to define an isomorphism [pw] of the complex vector space of quantum states onto *PW* by sending each quantum state *F* to the image by v of the section corresponding to $F \circ (6)$.

The map v will be defined in the general case by immersion of our line bundle in a trivial vector bundle, but this is unnecessary in the cases where the line bundle is trivial. For example, if $(C_\alpha)_0$ is trivial, the map

$$\sum: \quad (X, (A, b)(\tilde{\mathcal{B}}_\alpha)_0) \in R_4 \times \frac{\tilde{\mathcal{B}}_0}{(\tilde{\mathcal{B}}_\alpha)_0} \to (X, (A, b, 0, 0, 0)\ Ker\tilde{C}'_\alpha) \in R_4 \times \frac{\tilde{\mathcal{B}}_0 \times \mathbb{R}}{Ker\tilde{C}'_\alpha}$$

is a global section of (12) (if, moreover, $\alpha_{11} = 0$, Σ is also bijective). Hence, both the principal and the line bundles are trivial. The sections of the line bundle can be identified in a canonical way with vector-valued functions on the base space. More precisely, the map defined by sending the section corresponding to the pseudotensorial function F to the function $F \circ \Sigma$ is an isomorphism from the complex vector space of sections onto the complex vector space of complex-valued functions on $\mathbb{R}_4 \times (\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0)$.

Then, in the case where $(C_\alpha)_0$ is trivial, [pw] sends the quantum state whose image by [ps] is f to the complex-valued function on $\mathbb{R}_4 \times \tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$, Ψ_f, given by

$$\begin{aligned} &\Psi_f((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\ &\quad = f'((A, b)(\tilde{\mathcal{B}}_\alpha)_0)\ e^{-2\pi i \langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle} \end{aligned} \tag{5.11}$$

where f' is given by

$$f'((A, b)(\tilde{\mathcal{B}}_\alpha)_0) = f((A, b, 0, 0, 0)\ Ker\tilde{C}'_\alpha)$$

We thus see that in this case the prewave functions are functions of the form (5.11) with f' arbitrary.

In the case where $(C_\alpha)_0$ is not trivial, a way is given in Remark 5.1 of ref. 8 to immerse the line bundle in a trivial bundle. The idea is the following.

If ρ is a representation of a Lie group G in a finite-dimensional complex vector space L, ρ induces in a canonical way an action in the projective space (the differentiable manifold of the one-dimensional complex subspaces) $P(L)$ of L.

Let $L^* = L - \{0\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$. If $z \in L^*$, we denote by $[z] \in P(L)$ the one-dimensional subspace containing z.

Let $z_0 \in L^*$, G_{z_0} be the isotropy subgroup at z_0, and $G_{[z_0]}$ be the isotropy subgroup at $[z_0]$ We obviously have $G_{z_0} \in G_{[z_0]}$.

Now, let H be a closed subgroup of G such that $G_{z_0} \subset H \subset G_{[z_0]}$. For all $h \in H$ there exists $k(h) \in \mathbb{C}$ such that $\rho(h)z_0 = k(h)z_0$. The map k: $h \in H \rightarrow k(h) \in \mathbb{C}^*$ is a Lie group homomorphism whose kernel is G_{z_0}.

Then H/G_{z_0} is a Lie group and $(G/G_{z_0})(G/H, H/G_{z_0})$ is a principal fiber bundle. On the other hand, k gives us an action of the structural group H/G_{z_0} on $\mathbb{C}$ by means of $hG_{z_0} * c = k(h)c$ for all $h \in H$, $c \in \mathbb{C}$. The immersion of the associated line bundle $(G/G_{z_0}) \times_{(H/G_{z_0})} \mathbb{C}$ into the trivial bundle π_1: $(G/H) \times L \rightarrow G/H$ is defined by sending $[gG_{z_0}, c]_{(H/G_{z_0})}$ to $(gH, c\rho(g) \cdot z_0)$.

This map is an injective homomorphism of vector bundles which enables us to define an injective homomorphism from the complex vector space of sections of the former into the complex vector space of L-valued functions on G/H as follows: if f is a pseudotensorial function on G/G_{z_0}, the above map transforms the corresponding section of the associated line bundle into a section of π_1 whose composition with the canonical projection on L is the L-valued function on G/H, ϕ_f, given by $\phi_f(gH) = f(gG_{z_0})\ \rho(g) \cdot z_0$.

In order to apply this construction to our case, we assume that there exists a representation ρ of $\mathcal{B}_0 \times \mathbb{R}$ in a finite-dimensional vector space L and $z_0 \in L$ such that:

1. $\rho(g)z_0 = \tilde{C}'_\alpha(g)z_0$, $\forall g \in (\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$.
2. The isotropy subgroup at z_0, $(\mathcal{B}_0 \times \mathbb{R})_{z_0}$, is contained in $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$.

Under these circumstances, we say that (ρ, L, z_0) is a *trivialization* of $\tilde{C}'_\alpha$, and we have $(\mathcal{B}_0 \times \mathbb{R})_{z_0} = Ker\tilde{C}'_\alpha \subset (\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R} \subset (\mathcal{B}_0 \times \mathbb{R})_{[z_0]}$. Then, if we denote $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$ by H, the homomorphism k of the preceding discussion coincides with $\tilde{C}'_\alpha$.

The map [pw] is defined in this case as sending the quantum state F to the L-valued function Ψ_f, where $f = [\mathrm{ps}](F)$, given by

$$\begin{aligned}
&\Psi_f((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\
&\quad = W_f((I, 0, x, t, 0), (A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha)\rho(A, b, 0, 0, a) \cdot z_0 \\
&\quad = f((A, b, 0, 0, a)\ Ker\tilde{C}'_\alpha)\ e^{-2\pi i\langle (x,t), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}\ \rho(A, b, 0, 0, a) \cdot z_0 \qquad (5.12)
\end{aligned}$$

for all $(I, 0, x, t, 0) \in R_4$, $(A, b, 0, 0, a) \in \mathcal{B}_0 \times \mathbb{R}$.

The prewave functions in this case are thus functions having the form (5.12) with f pseudotensorial in Π.

The complex vector space of quantum states is a space of representation for $\tilde{\mathcal{B}}$: to each $g \in \tilde{\mathcal{B}}$ there corresponds an isomorphism that sends each quantum state F considered as a pseudotensorial function on $\tilde{\mathcal{B}}\ /Ker\tilde{C}_\alpha$ to $F \circ g^{-1}$, where g^{-1} means the diffeomorphism of $\tilde{\mathcal{B}}/Ker\tilde{C}_\alpha$ canonically associated to g. This representation then translates by means of [ps] (resp. [pw]) to a equivalent representation on PS (resp. PW), which will be denoted by ρ_{ps} (resp. ρ_{pw}). We now proceed to describe these representations.

If $g = (A', b', x', t', a')$, F is a quantum state and W_f its pullback by (6), the pullback of $F \circ g^{-1}$ is $W_f \circ \Phi_{g^{-1}}$, where $\Phi_{g^{-1}}$ is the diffeomorphism of $R_4 \times ((\mathcal{B}_0 \times \mathbb{R})/KerC'_\alpha)$ associated to g^{-1} by the action Φ given by equation (5.1) in the case where $L = Ker\tilde{C}_\alpha$.

We have

$$\begin{aligned}
&W_f \circ \Phi_{g^{-1}}((I, 0, x, t, 0), (A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)\\
&\quad = W_f(g^{-1} * (I, 0, x, t, 0), \nu(g^{-1}, (I, 0, x, t, 0))(A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)\\
&\quad = W_f(g^{-1} * (I, 0, x, t, 0), (A'^{-1}, -A'^{-1}b', 0, 0, -a' - b'^2t'/2 + \langle b', x'\rangle\\
&\qquad - {}^t(A'^{-1}b')(A'^{-1}x - A'^{-1}b't/2))(A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)\\
&\quad = W_f(g^{-1} * (I, 0, x, t, 0), (A'^{-1}, -A'^{-1}b', 0, 0, -a' - b'^2\, t'/2 + \langle b', x'\rangle\\
&\qquad + b'^2t/2 - {}^tb'x)(A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)\\
&\quad = f(A'^{-1}, A, A'^{-1}(b - b'), 0, 0, a - a' + b'^2(t - t')/2\\
&\qquad + \langle b', x' - x\rangle)\, Ker\tilde{C}'_\alpha)\; e^{-2\pi i\langle g^{-1}*(x,t), P((A',b')^{-1}(A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}\\
&\quad = f(A'^{-1}A, A'^{-1}(b - b'), 0, 0, 0)\, Ker\tilde{C}'_\alpha)\; e^{-2\pi i(a - a' + \langle b', x' - x + b'(t-t')/2\rangle)\alpha_{11}}\\
&\qquad \times e^{-2\pi i\langle g^{-1}*(x,t), P((A',b')^{-1}(A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}\\
&\quad = f((A', b', 0, 0, 0)^{-1}(A, b, 0, 0, 0)\, Ker\tilde{C}'_\alpha\\
&\qquad \times e^{-2\pi i(a - a' + (\langle x - x', t - t'), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle)}\\
&\quad = f((A', b', 0, 0, a')^{-1}(A, b, 0, 0, 0)\, Ker\tilde{C}'_\alpha)\; e^{-2\pi i\langle (x - x', t - t'), P((A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}
\end{aligned}$$

Then we see that

$$\rho_{ps}(g)(f) = e^{2\pi i\langle (x',t'), P\circ\pi\rangle}\, f \circ (A', b', 0, 0, a')^{-1} \tag{5.13}$$

On the other hand, for any prewave function Ψ_f we must have

$$\rho_{pw}(g)(\Psi_f) = \Psi_{\rho_{ps}(g)(f)}$$

so that, of course, (5.13) also determines ρ_{pw}. Now we shall give a formula that determines $\rho_{pw}(g)(\Psi_f)$ more directly in terms of Ψ_f.

Let us first consider the case where $(C_\alpha)_0$ is not trivial and (ρ, L, z_0) is a trivialization of $\tilde{C}'_\alpha$. One of our preceding computations leads to

$$
\begin{aligned}
&(\rho_{pw}(g)((\Psi_f))(I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\
&\quad = f(A'^{-1}A, A'^{-1}(b - b'), 0, 0, 0)\, Ker\tilde{C}'_\alpha) \\
&\qquad \times e^{-2\pi i(\langle g^{-1} * (x,t), P((A',b')^{-1}(A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle - a' + \langle b', x' - x + b'(t-t')/2\rangle)\alpha_{11})} \\
&\qquad \times \rho(A, b, 0, 0, 0) \cdot z_0
\end{aligned}
$$

On the other hand, if we denote by Φ'_g the diffeomorphism of $R_4 \times (\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0)$ corresponding to g by the action given by (5.1) in the case where $L = (\tilde{\mathcal{B}}_\alpha)_0 \times_r \mathbb{R}^5$, we have

$$
\begin{aligned}
&\Psi_f \circ \Phi'^{-1}_g((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\
&\quad \simeq \Psi_f \circ \Phi'^{-1}_g((I, 0, x, t, 0), (A, b, 0, 0, 0)(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}) \\
&\quad = \Psi_f(g^{-1} * (I, 0, x, t, 0), \\
&(A'^{-1}A, A'^{-1}(b - b'), 0, 0, -a' + b'^2(t - t')/2 \\
&\qquad + \langle b', x' - x\rangle)(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}) \\
&\quad \simeq \Psi_f(g^{-1} * (I, 0, x, t, 0), (A', b')^{-1}(A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\
&\quad = f((A'^{-1}A, A'^{-1}(b - b'), 0, 0, 0)\, Ker\tilde{C}'_\alpha) \\
&\qquad \times e^{-2\pi i\langle g^{-1} * (x,t), P((A',b')^{-1}(A,b)(\tilde{\mathcal{B}}_\alpha)_0)\rangle}\, \rho((A', b', 0, 0, 0)^{-1}(A, b, 0, 0, 0)) \cdot z_0 \\
&\quad = \rho(A', b', 0, 0, 0)^{-1}\, e^{2\pi i\alpha_{11}(-a' + b'^2(t-t')/2 + \langle b', x' - x\rangle)} \\
&\qquad \times ((\rho_{pw}(g)(\Psi_f))((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0))
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(\rho_{pw}(g)(\Psi_f))((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\
&\quad = e^{2\pi i\alpha_{11}(a' + b'^2(t'-t)/2 + \langle b', x - x'\rangle)}\, \rho(A', b', 0, 0, 0) \\
&\qquad \times (\Psi_f \circ \Phi'^{-1}_g((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0))
\end{aligned}
$$

In the case where $(C_\alpha)_0$ is trivial, if Ψ_f is given by (5.11) and $g = (A', b', x', t', a')$, we have

$$
\begin{aligned}
&(\rho_{pw}(g)(\Psi_f))((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\
&\quad = e^{2\pi i\alpha_{11}(a' + \langle b', x - x' + b'(t-t')/2\rangle)}\, \Psi_f \circ \Phi'^{-1}_g((I, 0, x, t, 0), (A, b)(\tilde{\mathcal{B}}_\alpha)_0)
\end{aligned}
$$

Up to this moment, we have made no topological restriction on the class of quantum states to be considered. Many choices are possible that enable us to define wave functions associated to the corresponding quantum states. Our choice in the present paper is the following: we consider only quantum states corresponding to prewave functions Ψ_f such that f is continuous with compact support.

Then, the Hermitian line bundle nature of some of the geometrical objects involved enables us to define in a canonical way a Hermitian product of quantum states and, as a consequence, of prewave functions. In fact, let F and F' be quantum states, σ_F and $\sigma_{F'}$ the associated sections of the Hermitian line bundle, $f = [\mathrm{ps}](F)$ and $f' = [\mathrm{ps}](F')$ (f and f' continuous with compact support). Then the Hermitian structure associates to the pair of quantum states the function

$$\begin{aligned}
&\langle\sigma_F, \sigma_{F'}\rangle((A, b)(\tilde{\mathcal{B}}_\alpha)_0) \\
&\quad = \overline{F((A, b, 0, 0, a)\, Ker\tilde{C}_\alpha)}\, F'((A, b, 0, 0, a)\, Ker\tilde{C}_\alpha) \\
&\quad = \overline{f((A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)}\, f'((A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)
\end{aligned}$$

for all $(A, b, 0, 0, a) \in \mathcal{B}_0 \times \mathbb{R}$, which leads us to define the Hermitian product of prewave functions

$$\langle\Psi_f, \Psi_{f'}\rangle = \int_{\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0} \bar{f}f'\omega$$

where ω is an invariant volume element on $\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$ and $\bar{f}f'$ is the function on $\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$ given by

$$\bar{f}f'((A, b)(\tilde{\mathcal{B}}_\alpha)_0) = \overline{f((A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)}\, f'((A, b, 0, 0, a)\, Ker\tilde{C}'_\alpha)$$

for all $(A, b, 0, 0, a) \in \mathcal{B}_0 \times \mathbb{R}$.

It follows from equation (5.13) that the representation is unitary for this Hermitian product.

This Hermitian product can be expressed in terms of the prewave functions themselves as follows.

If $(C_\alpha)_0$ is trivial, we obviously have

$$\langle\Psi_f, \Psi_{f'}\rangle = \int_{\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0} \overline{\Psi_f}\, \Psi_{f'}\omega \tag{5.14}$$

where $\overline{\Psi_f}\Psi_{f'}$ is the function on $\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$ given by

$$\overline{\Psi_f}\Psi_{f'}((A, b)(\tilde{\mathcal{B}}_\alpha)_0) = \overline{\Psi_f(H, (A, b)(\tilde{\mathcal{B}}_\alpha)_0)}\, \Psi_{f'}\, (H, (A, b)(\tilde{\mathcal{B}}_\alpha)_0)$$

for all $(A, b) \in (\tilde{\mathcal{B}}_\alpha)_0$, $H \in R_4$.

Now, let us assume that $(C_\alpha)_0$ is not trivial and (ρ, L, z_0) is a trivialization of $\tilde{C}'_\alpha$. Since the isotropy subgroup of $\mathcal{B}_0 \times \mathbb{R}$ at z_0 is $Ker\tilde{C}'_\alpha$, we have a canonical inmersion of $\mathcal{B}_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha$ into L whose image is the orbit of z_0, which will be denoted by $\mathcal{P}$. We identify $\mathcal{B}_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha$ to $\mathcal{P}$ by means of this immersion, but the topology and differentiable structure we consider is that of homogeneous space. The canonical projection of $\mathcal{B}_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha$ onto $\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$ becomes a map r of $\mathcal{P}$ onto $\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$. The pseudotensorial functions on $\mathcal{B}_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha$ become complex-valued functions on $\mathcal{P}$; in fact, they correspond to the functions which are homogeneous of degree -1 under multiplication by elements of $\tilde{C}'_\alpha((\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}) \subset \mathbb{S}^1$. These functions will be called α-homogeneous of degree -1. The α-homogeneous functions of degree T are defined in a similar way.

Let Φ be a sesquilinear form on L which does not vanish on $\mathcal{P}$. We define

$$\Psi_f\,\Phi\,\Psi_{f'}\colon\quad (A,b)(\tilde{\mathcal{B}}_\alpha)_0 \in \tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$$
$$\mapsto \frac{\Phi(\Psi_f(H,(A,b)(\tilde{\mathcal{B}}_\alpha)_0),\Psi_{f'}(H,(A,b)(\tilde{\mathcal{B}}_\alpha)_0))}{\Phi(z,z)}$$

where z is arbitrary in $\mathbf{r}^{-1}((A,b)(\tilde{\mathcal{B}}_\alpha)_0)$ and H is arbitrary in R^4. Thus

$$\langle \Psi_f, \Psi_{f'}\rangle = \int_{\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0} \Psi_f\Phi\Psi_{f'}\omega \tag{5.15}$$

Finally, we now define the wave functions. To each prewave function Ψ_f we associate a function $\tilde{\Psi}_f$ defined on $\mathbb{R}^4$ with values in the same space as Ψ_f by means of

$$\tilde{\Psi}_f(x,t) = \int_{\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0} \Psi_f((I,0,x,t,0),\cdot)\omega$$

Since we only consider the case where f is continuous with compact support, the $\tilde{\Phi}_f$ we obtain is analytic.

The functions $\tilde{\Phi}_f$ are called *wave functions* and their construction can be interpreted as follows. If one thinks of the prewave function Ψ_f as giving an "amplitude of probability" $\Psi_f((I,0,x,t,0),(A,b)(\tilde{\mathcal{B}}_\alpha)_0)$ for each event (x,t) and each class of movements passing trough (x,t), $(A,b)(\tilde{\mathcal{B}}_\alpha)_0$, $\tilde{\Psi}_f$ associates to (x,t) the "sum" of the amplitudes of probability corresponding to all the classes of movements passing trough (x,t).

If the correspondence $f \mapsto \tilde{\Psi}_f$ is injective, one obtains an action ρ_w of $\tilde{\mathcal{B}}$ on wave functions by means of

$$\rho_w(g)(\tilde{\Psi}_f) = \tilde{\Psi}_{\rho_{ps}(g)(f)} \tag{5.16}$$

Injectivity is not clear but, as a consequence of the invariance of ω, we have

$$\tilde{\Psi}_{\rho_{ps}(g)(f)} = \rho(A',b',0,0,0)\,e^{2\pi i\alpha_{11}(a'+\langle b',x-x'+b'(t'-t)/2\rangle)}\,\tilde{\Psi}_f\circ g^{-1} \tag{5.17}$$

for all $g = (A',b',x',t',a') \in \tilde{\mathcal{B}}$, where g^{-1} on the right-hand side stands for the diffeomorphism of $\mathbb{R}^4$ corresponding to g and x (resp. t) is the function on $\mathbb{R}^4$ whose value at (c^1, c^2, c^3, c^4) is (c^1, c^2, c^3) (resp. c^4). Notice that (5.17) is valid under the hypothesis that $(C_\alpha)_0$ is not trivial and (ρ, L, z_0) is a trivialization of $\tilde{C}'_\alpha$. If $(C_\alpha)_0$ is trivial, a formula similar to (5.17) is valid, with the factor $\rho(A', b', 0, 0, 0)$ deleted.

This entails in particular that $\tilde{\Psi}_{\rho_{ps}(g)(f)}$ only depends on $\tilde{\Psi}_f$ and g, so that (5.16) defines ρ_w, which, as a consequence of (5.16), is an action.

The differential of ρ_w, $d\rho_w$, is the map from the Lie algebra of $\tilde{\mathcal{B}}$ into the Lie algebra of endomorphisms of the space of wave functions defined by

$$(d\rho_w(X)(\tilde{\Psi}_f))(H) = \frac{d}{ds}\bigg|_0 (\rho_w(\operatorname{Exp} sX)(\tilde{\Psi}_f))(H)$$

for all $X \in \underline{\tilde{\mathcal{B}}}$, $H \in \mathbb{R}^4$.

If $X = (dq^{-1}(\hat{\eta}), \beta, \gamma, \delta, \kappa)$, where $\eta, \beta, \gamma \in \mathbb{R}^3$, $\delta, \alpha \in \mathbb{R}$, a straightforward computation proves that

$$\begin{aligned}(d\rho_w(X)(\tilde{\Psi}_f)) = {} & d\rho(dq^{-1}(\hat{\eta}), \beta, 0, 0, 0) \\ & \circ \tilde{\Psi}_f + 2\pi i \alpha_{11}(\kappa + \langle \beta, x \rangle)\tilde{\Psi}_f + X_{\mathbb{R}^4} \cdot \tilde{\Psi}_f \end{aligned} \tag{5.18}$$

where $X_{\mathbb{R}^4}$ is the vector field on $\mathbb{R}^4$ whose flow is composed of the diffeomorphisms associated by the action on $\mathbb{R}^4$ to $\{\operatorname{Exp}(-sX) \colon s \in \mathbb{R}\}$. In the case where $(C_\alpha)_0$ is trivial, the formula obtained from the preceding one by deleting the term in ρ is valid.

Each $X \in \underline{\tilde{\mathcal{B}}}$ is a function on the state space of the particle corresponding to α, and, in that sense, is a classical dynamical variable. The *quantum operator* associated to X is

$$\hat{X} = \frac{i}{2\pi} d\rho_w(X)$$

In the case of the canonical dynamical variables, one obtains

$$\tilde{l}_k = \frac{1}{2\pi i}\left(d\rho\left(\frac{i\sigma_k}{2}, 0, 0, 0, 0\right) + \sum_{j,r=1}^{3} \epsilon_{kjr}\, x^j \frac{\partial}{\partial x^r}\right)$$

$$\tilde{g}_k = \frac{1}{2\pi i}\left(d\rho(0, e_k, 0, 0, 0) + 2\pi i \alpha_{11} x^k - t\frac{\partial}{\partial x^k}\right)$$

$$\tilde{p}_k = \frac{1}{2\pi i}\frac{\partial}{\partial x^k}$$

$$\hat{E} = \frac{i}{2\pi}\frac{\partial}{\partial t}$$

$$\underline{\hat{m}} = \alpha_{11}$$

where ϵ_{kjr} are the components of an antisymmetric tensor such that $\epsilon_{123} = 1$.

The following sections study the wave functions that correspond to several kinds of particles.

6. SCHRÖDINGER EQUATION

Let us consider a particle whose movement space is the coadjoint orbit of the form

$$\alpha = {}^t(dQ)(\tau \overset{*}{Z}{}^{10} + \mu \overset{*}{Z}{}^{11}), \qquad \mu \neq 0$$

The isotropy subgroup at α in this case is

$$\tilde{\mathcal{B}}_\alpha = \{(U, 0, 0, e, a): U \in SU(2), e, a \in \mathbb{R}\}$$

This group is connected and the elements of its fundamental group are given in an obvious way by the elements of the fundamental group of SU(2). The integrals of α on these curves are all zero so that the cohomology class defined by α on $\tilde{\mathcal{B}}_\alpha$ is zero. It follows that α is $\mathbb{R}$-quantizable [7] and the homomorphism C'_α of $\tilde{\mathcal{B}}_\alpha$ onto $\mathbb{R}$ whose differential is α can be computed as follows.

For each $(U, 0, 0, e, a)$ in $\tilde{\mathcal{B}}_\alpha$ one consider a curve γ in $\tilde{\mathcal{B}}_\alpha$ beginning at the identity element and ending at $(U, 0, 0, e, a)$. Then

$$C'_\alpha(U, 0, 0, e, a) = \int_\gamma \alpha$$

One can take γ as being the product (in the homotopy theory sense) of a curve of the form $\gamma_1(t) = (\delta(t), 0, 0, 0, 0)$, where δ is a curve from I to U in SU(2), by the curve $\gamma_2(t) = (U, 0, 0, te, ta)$, and one obtains

$$C'_\alpha(U, 0, 0, e, a) = \tau e + \mu a$$

The homomorphism C_α of $\tilde{\mathcal{B}}_\alpha$ onto $\mathbb{S}^1$ whose differential is α is thus given by

$$C_\alpha(U, 0, 0, e, a) = e^{2\pi i(\tau e + \mu a)}$$

Then $(\tilde{\mathcal{B}}_\alpha)_0 = \{(U, 0): U \in \mathrm{SU}(2)\}$ and $(C_\alpha)_0$ is trivial, so that we need no trivialization to give the wave functions in this case.

We shall give a volume element on $\mathcal{B}_0/(\tilde{\mathcal{B}}_\alpha)_0$.

Let us consider the action of $\tilde{\mathcal{B}}_0$ on $\mathbb{R}^3$ given by

$$(U, b) * x = q(U)x + b$$

The isotrropy subgroup at 0 is $(\tilde{\mathcal{B}}_\alpha)_0$, so that $\tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0$ can be identified with $\mathbb{R}^3$ by means of the diffeomorphism

$$(U, b)(\tilde{\mathcal{B}}_\alpha)_0 \in \tilde{\mathcal{B}}_0/(\tilde{\mathcal{B}}_\alpha)_0 \to b \in \mathbb{R}^3$$

Since $det(q(U)) = 1$ for all $U \in \mathrm{SU}(2)$, the usual volume element on $\mathbb{R}^3$ is invariant.

With this identification the function P becomes

$$P(\mathrm{y}) = \begin{pmatrix} -\mu y \\ \tau + \mu y^2/2 \end{pmatrix}$$

As a consequence the wave functions are

$$\tilde{\Psi}_f(x, t) = \int_{\mathbb{R}^3} e^{2\pi i(\mu\langle x,y\rangle - t(\tau+\mu y^2/2))}\, d^3y$$

where f is a compactly supported function on $\mathbb{R}^3$.

A straightforward computation proves that these functions are solutions of the equation

$$\frac{i}{2\pi}\frac{\partial}{\partial t}\tilde{\Psi}_f(x, t) = \tau\tilde{\Psi}_f(x, t) - \frac{1}{(2\pi)^2}\frac{1}{2\mu}\Delta\tilde{\Psi}_f(x, t) \tag{6.1}$$

Notice that μ can take positive or negative values. For positive μ the preceding equation is the Schrödinger equation with a constant potential τ.

For a pair (τ, μ) with negative μ, one obtains an equation whose solutions are the complex conjugate of the solutions of the Schrödinger equation corresponding to $(-\tau, |\mu|)$.

The interior product of wave functions is given by (5.14).

7. PAULI EQUATIONS

Now we consider a particle whose movement space is the coadjoint orbit of

$$\alpha = {}^t(dQ)(s\overset{*}{Z}{}^3 + \tau\overset{*}{Z}{}^{10} + \mu\overset{*}{Z}{}^{11})$$

where $s \neq 0$, $\mu \neq 0$.

In this case we have

$$\tilde{\mathcal{B}}_\alpha = \left\{\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0, 0, e, a\right) : z \in \mathbb{S}^1, e, a \in \mathbb{R}\right\}$$

The group $\tilde{\mathcal{B}}_\alpha$ is connected so that the orbit is quantizable if and only if the cohomology class defined by α, $[\alpha]$, is in $H^1(\tilde{\mathcal{B}}_\alpha, \mathbb{Z})$ [7]. This condition holds if and only if $\int_\gamma \alpha$ is integral for all γ in a system of generators of the fundamental group of $\tilde{\mathcal{B}}_\alpha$. But the fundamental group of $\tilde{\mathcal{B}}_\alpha$ is generated by the curve given by

$$\gamma(t) = \begin{pmatrix} e^{2\pi it} & 0 \\ 0 & e^{-2\pi it} \end{pmatrix}$$

for all $t \in [0, 1]$.

We have $\int_\gamma \alpha = -4\pi s$, so that *the orbit is quantizable if and only if* $s = Z/4\pi$ *for some integral* Z. When this condition holds the integer Z is the period of the Reeb vector field of the contact manifold and the homomorphism C_α is given by

$$C_\alpha(g) = e^{2\pi i \int_\delta \alpha}$$

where δ is a curve beginning at the identity element and ending at g.

Let $z = e^{2\pi ir}$, $r, e, a \in \mathbb{R}$. in order to evaluate

$$C_\alpha\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0, 0, e, a\right)$$

one can consider the curve

$$\delta(t) = \left(\begin{pmatrix} e^{2\pi itr} & 0 \\ 0 & \overline{e^{-2\pi itr}} \end{pmatrix}, 0, 0, te, ta\right)$$

and one obtains $\int_\delta \alpha = -4\pi rs + \tau e + \mu a$ so that

$$C_\alpha\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0, 0, e, a\right) = z^{-4\pi s} e^{2\pi i(\tau e + \mu a)}$$

Then

$$(\tilde{\mathcal{B}}_\alpha)_0 = \left\{\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0\right) : z \in \mathbb{S}^1\right\}$$

$$(C_\alpha)_0\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0\right) = z^{-4\pi s}$$

$$(\mathcal{B}_\alpha)_0 \times_r \mathbb{R}^5 = \left\{\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0, c, e, a\right) : z \in \mathbb{S}^1, c \in \mathbb{R}^3, e, a, \in \mathbb{R}\right\}$$

$$\tilde{C}_\alpha\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0, c, e, a\right) = z^{-4\pi s} e^{2\pi i(\tau e + \mu a)}$$

$$(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R} = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0, 0, 0, a \right) : z \in \mathbb{S}^1, a \in \mathbb{R} \right\}$$

$$\tilde{C}'_\alpha \left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, 0, 0, 0, a \right) = z^{-4\pi s} e^{2\pi i \mu a}$$

Now let us restrict ourselves to the case where $-4\pi s = 1$.

In this case $(\rho, z_0, \mathbb{C}_4)$ is a trivialization of $\tilde{C}'_\alpha$, where $z_0 = {}^t(1, 0, 0, 0)$ and ρ is the representation of $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$ in $\mathbb{C}^4$ given by

$$\rho(U, b, 0, 0, a) = e^{2\pi i \mu a} \begin{pmatrix} U & 0 \\ -\frac{1}{2} h(b) U & U \end{pmatrix} \in GL(4, \mathbb{C})$$

This trivialization enables us to give the wave functions in terms of homogeneous of degree -1 functions on the orbit of z_0 under ρ, identified with $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha$. We will identify these spaces with a well-known manifold, which enables us to describe the construction in a more geometrical way.

We define an action of $\mathcal{B}_0 \times \mathbb{R}$ on $S_3 \times \mathbb{R}^3$, where S^3 is the usual unit sphere in $\mathbb{C}^2$, by means of

$$(U, b, 0, 0, a) * (z, r) = (e^{2\pi i \mu a} Uz, q(U)r + b)$$

This is a transitive action and the isotropy subgroup at $({}^t(1, 0), 0)$ is $Ker\tilde{C}'_\alpha$. As a consequence, the map

$$\lambda: \quad (U, b, 0, 0, a)\, Ker\tilde{C}'_\alpha \in \frac{\mathcal{B}_0 \times \mathbb{R}}{Ker\tilde{C}'_\alpha} \to (U, b, 0, 0, a)$$

$$* \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \right) \in S^3 \times \mathbb{R}^3$$

is a diffeomorphism. If one identifies these manifolds by means of λ, one sees by direct computation that the bundle action of $\mathbb{S}^1$ on $\mathcal{B}_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha$ becomes the usual product by elements of $\mathbb{S}^1$ in the factor S^3.

Now we consider an action of $\mathcal{B}_0 \times \mathbb{R}$ in $P_1(\mathbb{C}) \times \mathbb{R}^3$, where $P_1(\mathbb{C})$ is the projective space corresponding to $\mathbb{C}^2$, i.e., the manifold of complex 1-dimensional subspaces of $\mathbb{C}^2$, by means of

$$(U, b, 0, 0, a) * ([z], r) = ([Uz], q(U)r + b)$$

This action is transitive (since the preceding one is) and the isotropy subgroup at $([{}^t(1, 0)], 0)$ is $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$. As a consequence, the map

$$\lambda': \quad (U, b, 0, 0, a)(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R} \in \frac{\mathcal{B}_0 \times \mathbb{R}}{(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}} \to (U, b, 0, 0, a)$$

$$* \left(\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right], 0 \right) \in P_1(\mathbb{C}) \times \mathbb{R}^3$$

is a diffeomorphism. If one identifies these manifolds by means of λ', the principal fiber bundle

$$\frac{\mathcal{B}_0 \times \mathbb{R}}{Ker\tilde{C}'_\alpha} \left(\frac{\mathcal{B}_0 \times \mathbb{R}}{(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}}, \mathbb{S}^1 \right)$$

becomes $(\mathbb{S}^3 \times \mathbb{R}^3)((P_1(\mathbb{C}) \times \mathbb{R}^3, \mathbb{S}^1)$, where the bundle projection maps each (z, r) to $([z], r)$, and the bundle action is given, as we have said above, by the usual product of elements of S^3 by modulus one complex numbers.

The 2-form on $\mathbb{C}^2$ given by

$$\nu_0 = \frac{(z^1 dz^2 - z^2 dz^1) \wedge \overline{(z^1 dz^2 - z^2 dz^1)}}{(z^* z)^2}$$

where z^1, z^2 are the canonical coordinates in $\mathbb{C}^2$ and $z^*z = |z^1|^2 + |z^2|^2$ is projectable in $P_1(\mathbb{C})$. The local expression of the projection ν in each of the two canonical coordinate systems of $P_1(\mathbb{C})$ is

$$\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}$$

The 5-form defined in $P_1(\mathbb{C}) \times \mathbb{R}^3$ by $\nu \wedge \omega$, where ω is the canonical volume element of $\mathbb{R}^3$, is a volume element left invariant by the action of $\mathcal{B}_0 \times \mathbb{R}$.

With the identifications we have done, the function P that was originally defined in $\mathcal{B}_0 \times \mathbb{R}/(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$, becomes a function on $P_1(\mathbb{C}) \times \mathbb{R}^3$. In a strict sense this function is $P \circ (\lambda')^{-1}$. But we have

$$(\lambda')^{-1}\left(\left[\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \right], r \right)$$

$$= \left(\begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} & \frac{-\overline{z^2}}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} & \frac{\overline{z^1}}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r, 0, 0, 0 \right) (\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$$

Then one sees that

$$P\left(\left[\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right], r\right) = \begin{pmatrix} -\mu r \\ \tau + \mu r^2/2 \end{pmatrix}$$

If f is a function on $S^3 \times \mathbb{R}^3$ homogeneous of degree -1 under multiplication by modulus one complex numbers in the first argument, the corresponding prewave function is given by

$$\begin{aligned}
&\Psi_f\left((I, 0, x, t, 0), \left[\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right], r\right) \\
&\quad = f\left(\begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r\right) \\
&\qquad \times e^{-2\pi i\langle (x,\, t), (-\mu r, \tau + \mu r^2/2)\rangle} \\
&\qquad \times \rho\left(\begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} & \dfrac{-\overline{z^2}}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} & \dfrac{\overline{z^1}}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r, 0, 0, 0\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\quad = f\left(\begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r\right) e^{-2\pi i\langle (x,t), (-\mu r, \tau + \mu r^2/2)\rangle} \\
&\qquad \times \begin{pmatrix} I \\ -\frac{1}{2}h(r) \end{pmatrix} \begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}
\end{aligned}$$

If f is continuous with compact support, the corresponding wave function $\tilde{\Psi}_f$ is obtained by integration with respect to $\nu \wedge \omega$. It is a function with values in $\mathbb{C}^4$. Each of its components satisfies equation (6.1).

Now let us denote

$$\Psi_f = \begin{pmatrix} \varphi_f \\ \chi_f \end{pmatrix} \tag{7.1}$$

and, accordingly

$$\tilde{\Psi}_f = \begin{pmatrix} \tilde{\varphi}_f \\ \tilde{\chi}_f \end{pmatrix}$$

where φ_f, χ_f, $\tilde{\varphi}_f$, $\tilde{\chi}_f$ take their values at $\mathbb{C}^2$.

Then we have

$$\tilde{\varphi}_f(x, t) = \int_{P_1(\mathbb{C})\times\mathbb{R}^3} f\left(\begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r\right) \times e^{-2\pi i\langle(x,t),(-\mu r,\tau+\mu r^2/2)\rangle} \begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix} \nu \wedge \omega \tag{7.3}$$

$$\tilde{\chi}_f(x, t) = -\frac{1}{2}\int_{P_1(\mathbb{C})\times\mathbb{R}^3} f\left(\begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r\right) \times e^{-2\pi i\langle(x,t),(-\mu r,\tau+\mu r^2/2)\rangle} h(r) \begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix} \nu \wedge \omega \tag{7.4}$$

and a straightforward computation proves that

$$\begin{aligned} E\tilde{\varphi}_f + (\sigma \cdot p)\,\tilde{\chi}_f &= 0 \\ (\sigma \cdot p)\tilde{\varphi}_f + 2\mu\tilde{\chi}_f &= 0 \end{aligned} \tag{7.5}$$

where

$$E = -\frac{1}{(2\pi)^2}\frac{1}{2\mu}\,\Delta$$

and

$$\sigma \cdot p = \sum_{j=1}^{3} \sigma^j \frac{1}{2\pi i}\frac{\partial}{\partial x^j}$$

In the case of a positive μ, these are the equations that appear in the Pauli theory of the nonrelativistic approach to the Dirac equation. Since the components of $\tilde{\varphi}_f$ and $\tilde{\chi}_f$ are components of $\tilde{\Psi}_f$, they are also solutions of the Schrödinger equation (6.1).

Now, let us determine the interior product in this case, according to (5.15).

The form

$$\Phi(Z, Z') = Z^* \gamma Z$$

where

$$\gamma = \begin{pmatrix} I & iI \\ -iI & 0 \end{pmatrix}$$

is sesquilinear. It is preserved by the representation ρ and its value at z_0 is 1. Hence its value at any point of $\mathcal{P}$ is 1.

Then the interior product is given by

$$\langle \Psi_f, \Psi_{f'} \rangle = \int_{P_1(\mathbb{C})\times\mathbb{R}^3} \Psi_f \gamma \Psi_{f'} \nu \wedge \omega \tag{7.6}$$

$$= \int_{P_1(\mathbb{C})\times\mathbb{R}^3} (\varphi_f^* \varphi_{f'} + i(\varphi_f^* \chi_{f'} - \chi_f^* \varphi_{f'}))\nu \wedge \omega$$

Now, let us consider the case where $4\pi s = 1$. A trivialization of $\tilde{C}'_\alpha$ is $(\rho', z_0, \mathbb{C}^4)$, where ρ' is the representation

$$\rho'(U, b, 0, 0, a) = e^{2\pi i \mu a} \begin{pmatrix} \overline{U} & 0 \\ \frac{1}{2} \overline{h(b) U} & \overline{U} \end{pmatrix} \in GL(4, \mathbb{C})$$

Also in this case we consider an action of $\mathcal{B}_0 \times \mathbb{R}$ on $S^3 \times \mathbb{R}^3$:

$$(U, b, 0, 0, a) * (z, r) = (e^{2\pi i \mu a}\, \overline{U}\, z, q(U) r + b)$$

and an action of $\mathcal{B}_0 \times \mathbb{R}$ on $P_1(\mathbb{C}) \times \mathbb{R}^3$:

$$(U, b, 0, 0, a) * ([z], r) = ([\overline{U} z], q(U) r + b)$$

The isotropy subgroup at $({}^t(1, 0), 0)$ (resp. $([{}^t(1, 0)], 0)$) is $Ker\tilde{C}'_\alpha$ (resp. $(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$) and both actions are transitive. Then we have a canonical diffeomorphism from $\mathcal{B}_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha$ (resp. $\mathcal{B}_0 \times \mathbb{R}/(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$) onto $S^3 \times \mathbb{R}^3$ (resp. $P_1(\mathbb{C}) \times \mathbb{R}^3$) defined as λ (resp. λ') above, but now * stands for the new action.

By means of these diffeomorphisms, the principal fiber bundle

$$(\mathcal{B}_0 \times \mathbb{R}/Ker\tilde{C}'_\alpha)(\mathcal{B}_0 \times \mathbb{R}/(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}, \mathbb{S}^1)$$

can be identified, also this time, with $(S^3 \times \mathbb{R}^3)(P_1(\mathbb{C}) \times \mathbb{R}^3, \mathbb{S}^1)$ with the same projection and bundle action as in the preceding case.

Now $(\lambda')^{-1}$ is given by

$$(\lambda')^{-1}\left(\left[\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right], r\right)$$

$$= \left(\begin{pmatrix} \frac{\bar{z}^1}{\sqrt{|z^1|^2 + |z^2|^2}} & \frac{-z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{\bar{z}^2}{\sqrt{|z^1|^2 + |z^2|^2}} & \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r, 0, 0, 0\right)(\tilde{\mathcal{B}}_\alpha)_0 \times \mathbb{R}$$

and one obtains also in this case

$$P\left(\left[\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right], r\right) = \begin{pmatrix} -\mu r \\ \tau + \mu r^2/2 \end{pmatrix}$$

If f is a complex-valued function on $S^3 \times \mathbb{R}^3$, homogeneous of degree -1 under multiplication in S^3 by modulus one complex numbers, the corresponding prewave function is given by

$$\Psi_f\left((I, 0, x, t, 0), \left[\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}\right], r\right)$$

$$= f\left(\begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r\right)$$

$$\times\ e^{-2\pi i\langle (x,t),(-\mu r, \tau + \mu r^2/2)\rangle}$$

$$\rho'\left(\begin{pmatrix} \frac{\bar{z}^1}{\sqrt{|z^1|^2 + |z^2|^2}} & \frac{-z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{\bar{z}^2}{\sqrt{|z^1|^2 + |z^2|^2}} & \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r, 0, 0, 0\right)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= f\left(\begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}, r\right) e^{-2\pi i\langle (x,t),(-\mu r, \tau + \mu r^2/2)\rangle}$$

$$\times \begin{pmatrix} I \\ \frac{1}{2}h(r) \end{pmatrix}\begin{pmatrix} \frac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \frac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix}$$

Then we define φ_f and χ_f by (7.1) and $\tilde{\varphi}_f$ and $\tilde{\chi}_f$ by (7.2) and we obtain

$$\tilde{\varphi}_f(x,\, t) = \int_{P_1(\mathbb{C})\times\mathbb{R}^3} f\left(\begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix},\, r\right)$$

$$\times\, e^{-2\pi i\langle (x,t),(-\mu r,\tau+\mu r^2/2)\rangle} \begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix} \nu \wedge \omega \tag{7.7}$$

$$\tilde{\chi}_f(x,\, t) = \frac{1}{2}\int_{P_1(\mathbb{C})\times\mathbb{R}^3} f\left(\begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix},\, r\right)$$

$$\times\, e^{-2\pi i\langle (x,t),(-\mu r,\tau+\mu r^2/2)\rangle}\, \overline{h(r)} \begin{pmatrix} \dfrac{z^1}{\sqrt{|z^1|^2 + |z^2|^2}} \\ \dfrac{z^2}{\sqrt{|z^1|^2 + |z^2|^2}} \end{pmatrix} \nu \wedge \omega \tag{7.8}$$

which are solutions of the system of equations

$$\begin{aligned} E\tilde{\varphi}_f + \overline{(\sigma \cdot p)}\tilde{\chi}_f &= 0 \\ \overline{(\sigma \cdot p)}\tilde{\varphi}_f + 2\mu\tilde{\chi}_f &= 0 \end{aligned} \tag{7.9}$$

where $\overline{(\sigma \cdot p)}$ is the complex conjugate operator of $\sigma \cdot p$.

Also by a straightforward computation one sees that the components of $\tilde{\varphi}_f$ and $\tilde{\chi}_f$ are solutions of the Schrödinger equation (6.1).

Notice that the complex conjugate of a solution of (7.5) whose components satisfy (6.1) is solution of (7.9), but its components do not satisfy (6.1).

With regard to the Hermitian product, one also can use here the sesquilinear form Φ and one obtains (7.6) also in this case.

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